A class of bivariate stochastic orderings, with applications in actuarial sciences

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Abstract

This paper is concerned with the bivariate extension of a wide class of univariate orderings said to be of convex-type. Attention is paid to random vectors valued in a rectangle or an orthant of the real plane. Various orderings used in probability and statistics (such as the stochastic dominance, the upper orthant order, the orthant convex order, the correlation order and the supermodular order) can be seen as particular cases. The practical applications of these orderings seem to be very promising, especially in actuarial sciences. ©1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Stochastic orderings among random vectors have a wide field of applications in probability and statistics; see, e.g., the book by Shaked and Shanthikumar (1994), especially Chapters 4 and 5, and the classified bibliography by Mosler and Scarsini (1993).

In actuarial sciences, multivariate orderings can be a very useful tool in order to take into account the possible dependence between the different risks the insurer faces or between the lifelengths involved in a multiple insurance contract. In particular, they can allow the actuary to measure the impact of this eventual dependence in tariffication. The reader is referred to the recent papers by Dhaene and Goovaerts (1996, 1997), as well as to the previous papers by Carrière and Chan (1986) and Norberg (1989).

Recently, Denuit and Lefèvre (1997a) and Denuit et al. (1998,1999b) proposed a wide class of univariate orderings of convex-type. Our aim in the present work is to introduce and investigate a new family of stochastic order relations among bivariate random vectors. This research follows two previous papers by Denuit and Lefèvre (1997b) and Denuit et al. (1999a) devoted to bivariate extensions of that kind. The first one dealt with product orderings and the second one with orderings among random vectors valued in a bidimensional grid of equally
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The paper is organized as follows. The bivariate stochastic order relations under interest are defined in Section 2. A brief review of their univariate counterparts is provided in order to motivate our definitions. Then, in Sections 3 and 4 the bidimensional versions are studied, including characterizations and properties. In Section 5, a special case of particular interest, namely the correlation order, is investigated. Various known results in the literature are extended there. Finally, in Section 6, a method to get bounds on various actuarial quantities is provided.

A few words on the notation and terminology used throughout the paper. In the remainder, we denote the random vector (X1, X2) by an underlined capital letter X and a point (x1, x2) of the real plane R2 by an underlined small letter x. The positive quadrant is denoted by R2+. The vector of ones, that is (1,1), will be denoted by 1; similarly, 2 = (2,2), and so on. Then, x ± y stands for (x1 ± y1, x2 ± y2). The space R2 is endowed with the usual componentwise partial order, i.e., x ≤ y if xi ≤ yi for i = 1, 2. We denote by S a continuum, that is a subinterval of the real line R. S may be open, half-open, or closed, finite or infinite; a and b represent the left and right endpoints of S (with the understanding that as soon as they are written, they are tacitly assumed to be finite). Analogously, S = S1 × S2 will represent a continuum, i.e. a rectangle, of the real plane R2. S may be open, half-open, or closed, finite or infinite; ai and bi are the left and right endpoints of Si, i = 1, 2, with the same convention as above, and [ai, bi] is the rectangle [ai, bi] × [a2, b2]. We recall that an upper orthant with vertex a is a subset of R2 of the form {x ∈ R2 | x ≥ a} for some fixed a ∈ R2, while the corresponding lower orthant is defined by {x ∈ R2 | x ≤ a}; they are denoted by [a, +∞] and [−∞, a], respectively. For y ∈ R, we put y+ = max(0, y), with the understanding that y+0 = 1 if y > 0 and 0 otherwise. Finally, we denote by N the set of the non-negative integers, and by N2 the cartesian product N × N.

2. Bivariate orderings of convex-type

2.1. Comparing random variables

2.1.1. Univariate integral stochastic orderings

Most univariate stochastic orderings ≤F are used in actuarial sciences to compare two random variables (two risks, say) X and Y valued in S ⊆ R are defined (or can be defined) by reference to a cone F of measurable functions φ: S → R by the equivalence

\[ X \preceq_F Y \iff E\phi(X) \leq E\phi(Y) \quad \text{for all functions } \phi \in \mathcal{F} \tag{2.1} \]

for which the expectations exist. Such orderings are referred to as integral stochastic orderings and obviously rely on the classical expected utility theory (by considering F as a class of (utility functions of) "rational" decision-makers). A rather general theory of the univariate integral stochastic orderings can be found in Müller (1997).

Let us now point out some remarkable particular cases of (2.1). The stochastic dominance ≤st is obtained with F = Fst, the class of the functions φ possessing a non-negative first derivative, while we get the stop-loss order ≤sl when F = Fsl, the class of the functions φ possessing non-negative first and second derivatives. Let us mention that not all stochastic order relations are integral stochastic orderings. For instance, the likelihood ratio order cannot be defined through (2.1) (see, e.g. Müller (1997)).
2.1.2. Generating cone of s-convex functions

Denuit et al. (1998) introduced various classes of integral stochastic orderings of convex-type for comparing continuous random variables. Specifically, given any positive integer \( s \), the \( s \)-convex ordering is generated by the class \( \mathcal{U}_{s-\text{cx}}^S \) of the regular \( s \)-convex functions on \( S \) which is defined by

\[
\mathcal{U}_{s-\text{cx}}^S = \left\{ \phi : S \to \mathbb{R} \mid \frac{d^s \phi}{dx^s} \geq 0 \text{ on } S \right\}.
\]  

(2.2)

The name comes from the fact that the functions in \( \mathcal{U}_{2-\text{cx}}^S \), i.e. the regular 2-convex functions, are convex in the usual sense (the term “regular” is added since all the functions with non-negative second derivative are convex but there exist convex functions that are not twice differentiable; the same phenomenon arises for the \( s \)-convex functions). Similarly, the class \( \mathcal{U}_{s-\text{cv}}^S \) of the regular \( s \)-concave functions on \( S \) is given by

\[
\mathcal{U}_{s-\text{cv}}^S = \left\{ \phi : S \to \mathbb{R} \mid (-1)^{s+1} \frac{d^s \phi}{dx^s} \geq 0 \text{ on } S \right\}.
\]

(2.3)

The appellation also originates in the fact that \( \mathcal{U}_{s-\text{cv}}^S \) contains usual concave functions. Finally, the classes \( \mathcal{U}_{s-\text{icx}}^S \) and \( \mathcal{U}_{s-\text{icv}}^S \) of the regular \( s \)-increasing convex and concave functions are, respectively, given by

\[
\mathcal{U}_{s-\text{icx}}^S = \bigcap_{k=1}^s \mathcal{U}_{k-\text{cx}}^S, \quad \text{and} \quad \mathcal{U}_{s-\text{icv}}^S = \bigcap_{k=1}^s \mathcal{U}_{k-\text{cv}}^S.
\]

(2.4)

The class \( \mathcal{U}_{s-\text{icx}}^S \) thus contains the functions with non-negative derivatives of degrees 1 to \( s \), while \( \mathcal{U}_{s-\text{icv}}^S \) contains the non-decreasing functions with derivatives of degrees 1 to \( s \) with alternating signs.

2.1.3. Univariate stochastic \( s \)-orderings

Now, given two random variables \( X \) and \( Y \) valued in \( S \), \( X \) is said to be smaller than \( Y \) in the \( s \)-convex (resp. \( s \)-concave, \( s \)-increasing convex and \( s \)-increasing concave) ordering, denoted by \( X \preceq_{s-\text{cx}}^S Y \) (resp. \( X \preceq_{s-\text{cv}}^S Y \), \( X \preceq_{s-\text{icx}}^S Y \) and \( X \preceq_{s-\text{icv}}^S Y \)), when (2.1) holds with \( F = \mathcal{U}_{s-\text{cx}}^S \) (resp. \( \mathcal{U}_{s-\text{cv}}^S \), \( \mathcal{U}_{s-\text{icx}}^S \) and \( \mathcal{U}_{s-\text{icv}}^S \)).

The ordering \( \preceq_{s-\text{icx}}^S \) is known in actuarial sciences as the stop-loss order of degree \( s - 1 \) (see, e.g. Goovaerts et al. (1990) and Hesselager (1996), Definition 1). Moreover, the ordering \( \preceq_{s-\text{icx}}^S \) viewed as a strengthening of \( \preceq_{s-\text{icx}}^\mathbb{R}^+ \) (obtained by requiring the equality of the \( s - 1 \) first moments of the random variables to be compared) has been previously considered, e.g., by Kaas et al. (1994) and Kaas and Hesselager (1995). The concave counterpart \( \preceq_{s-\text{icv}}^\mathbb{R}^+ \) is known in economics as the stochastic dominance of degree \( s \) (see, e.g., the review paper by Levy 1992).

Whereas we consider here random variables valued in a continuum, we mention that similar \( s \)-orderings have been studied by Denuit and Lefèvre (1997a) and Denuit et al. (1999b) to compare random variables valued in the grid \( D_n = \{0, 1, \ldots, n\} \) or in an arbitrary discrete set.

2.1.4. Restriction to the convex case

There exists a strong “duality” among the orderings \( \preceq_{s-\text{cx}}^S \) and \( \preceq_{s-\text{cv}}^S \), as well as among \( \preceq_{s-\text{icx}}^S \) and \( \preceq_{s-\text{icv}}^S \). This originates in the links between the corresponding generating cones of functions. It is directly seen from (2.2) and (2.3) that

\[
\phi \in \mathcal{U}_{s-\text{cv}}^S \iff \phi \in \mathcal{U}_{s-\text{cx}}^S \quad \text{if \( s \) is odd}, \quad \phi \in \mathcal{U}_{s-\text{cv}}^S \iff -\phi \in \mathcal{U}_{s-\text{cx}}^S \quad \text{if \( s \) is even},
\]

(2.5)

which results in

\[
X \preceq_{s-\text{cv}}^S Y \iff X \preceq_{s-\text{cx}}^S Y \quad \text{if \( s \) is odd}, \quad X \preceq_{s-\text{cv}}^S Y \iff Y \preceq_{s-\text{cx}}^S X \quad \text{if \( s \) is even}.
\]

(2.6)
On the other hand, we have from (2.4) that when \( \mathcal{S} = [a, b] \) or \( ] - \infty, b] \),
\[
\phi \in \mathcal{U}_{s-icv}^S \iff -\phi(b - \cdot) \in \mathcal{U}_{s-icx}^S,
\]
(2.7)
where \( \hat{S} = [b - \xi | \xi \in \mathcal{S}] \), while for \( \mathcal{S} = \mathbb{R} \),
\[
\phi \in \mathcal{U}_{s-icv}^R \iff -\phi(\cdot) \in \mathcal{U}_{s-icx}^R,
\]
(2.8)
so that (2.7) and (2.8), respectively, yield
\[
X \preceq_{s-icv}^S Y \iff b - Y \preceq_{s-icx}^S b - X, \quad \text{and} \quad X \preceq_{s-icv}^R Y \iff -Y \preceq_{s-icx}^R -X.
\]
(2.9)

To avoid problems of technical nature, in the \( s \)-increasing cases, it is preferable to consider the domain \( \mathcal{S} \) as bounded below (i.e., \( \mathcal{S} = [a, b] \) or \( [a, +\infty] \)). In this situation, one may safely assume that the test functions in \( \mathcal{U}_{s-icx}^S \) and \( \mathcal{U}_{s-icx}^R \) are non-negative (simply by substituting \( \phi(\cdot) - \phi(a) \) for \( \phi(\cdot) \)).

2.1.5. Limiting forms

Let us examine the limiting forms of \( \leq_{s-icv}^S \) and \( \leq_{s-icv}^S \) as \( s \to +\infty \). Hereafter, we will only deal with non-negative random variables \( (\mathcal{S} = \mathbb{R}^+) \).

We first consider \( \leq_{-icv}^{\infty-R} \). This integral stochastic ordering is generated through (2.1) with for \( \mathcal{F} \) the class
\[
\mathcal{U}_{-icv}^{\infty-R} = \left\{ \phi: \mathbb{R}^+ \to \mathbb{R} \mid (-1)^{k+1} \frac{d^k \phi}{dx^k} \geq 0 \text{ on } \mathbb{R}^+ \text{ for all } k \geq 1 \right\},
\]
(2.10)
i.e., a function \( \phi \) belongs to \( \mathcal{U}_{-icv}^{\infty-R} \) if, and only if, \( -\phi \) is completely monotone on \( \mathbb{R}^+ \). We point out below that the ordering \( \leq_{-icv}^{\infty-R} \) is equivalent to the classical Laplace transform ordering defined among non-negative random variables \( X \) and \( Y \) by
\[
X \preceq_{Lt} Y \iff Ee^{-cX} \geq Ee^{-cY} \text{ for all } c > 0;
\]
see, e.g. Shaked and Shanthikumar (1994), p. 95. Note that \( \leq_{Lt} \) is the integral stochastic ordering generated through (2.1) with \( \mathcal{F} = \mathcal{U}_{Lt}^{\infty-R} \equiv \{-e^{-cx}, c \in \mathbb{R}^+\} \).

**Proposition 2.1.** \( X \preceq_{-icv}^{\infty-R} Y \iff X \preceq_{Lt} Y \).

**Proof.** As \( \mathcal{U}_{Lt}^{\infty-R} \subset \mathcal{U}_{-icv}^{\infty-R} \), the “\( \implies \)”-part is obvious. For the converse, we know from Theorem 3.B.2 in Shaked and Shanthikumar (1994) that \( X \preceq_{Lt} Y \iff E\phi(X) \geq E\phi(Y) \) for all completely monotone functions \( \phi \) such that the expectations exist. Now, the operator \( \tau_- \) that, when applied to \( \phi \), gives the function \( -\phi \), is a bijection between \( \mathcal{U}_{-icv}^{\infty-R} \) and the class of the completely monotone functions, resulting in \( E\phi(Y) \geq E\phi(X) \) for any \( \phi \in \mathcal{U}_{-icv}^{\infty-R} \). \( \square \)

Now, \( \leq_{-icv}^{\infty-R} \) is the integral stochastic ordering generated through (2.1) with for \( \mathcal{F} \) the class
\[
\mathcal{U}_{-icv}^{\infty-R} = \left\{ \phi: \mathbb{R}^+ \to \mathbb{R} \mid \frac{d^k \phi}{dx^k} \geq 0 \text{ on } \mathbb{R}^+ \text{ for all } k \geq 1 \right\},
\]
(2.11)
which is the class of absolutely monotone functions on \( \mathbb{R}^+ \). We recall that the moment ordering (see, e.g. Shaked and Shanthikumar (1994), p. 103) is defined among non-negative random variables \( X \) and \( Y \) by
\[
X \preceq_{\text{moments}} Y \iff EX^k \leq EY^k \text{ for all positive integers } k,
\]
i.e., \( \preceq_{\text{moments}} \) is the integral stochastic ordering generated through (2.1) with \( \mathcal{F} = \{x^k, k \in \mathbb{N}\} \). Thus, it is clear that
\[
X \preceq_{-icv}^{\infty-R} Y \Rightarrow X \preceq_{\text{moments}} Y.
\]
(2.12)
It is possible to characterize the ordering \( \leq_{\text{icx}} \) through (2.1) with for \( \mathcal{F} \) the class of functions \( U_{\exp}^{\mathbb{R}^+} = \{ e^{cx}, c \in \mathbb{R}^+ \} \). The ordering associated with \( U_{\exp}^{\mathbb{R}^+} \) is sometimes called the exponential ordering (see Definition 3.1, p. 53 in Kaas et al. (1995)).

**Proposition 2.2.** \( X \leq_{\text{icx}} Y \) if, and only if, (2.1) holds with \( \mathcal{F} = U_{\exp}^{\mathbb{R}^+} \).

**Proof.** The “\( \Rightarrow \)”-part is obvious since \( U_{\exp}^{\mathbb{R}^+} \subseteq U_{\text{icx}}^{\mathbb{R}^+} \). For the converse, when \( \phi \in U_{\text{icx}}^{\mathbb{R}^+} \), we know that there exists a measure \( \mu \) on \( \mathbb{R}^+ \) such that

\[
\phi(x) = \int_{t=0}^{+\infty} e^{tx} d\mu(t), \quad x \in \mathbb{R}^+;
\]

see, e.g., Widder (1946), Theorem 12c p. 162. Thus, if (2.1) holds with \( \mathcal{F} = U_{\exp}^{\mathbb{R}^+} \), we have for any \( \phi \in U_{\text{icx}}^{\mathbb{R}^+} \) that

\[
E\phi(Y) - E\phi(X) = \int_{t=0}^{+\infty} E(e^{tY} - e^{tX}) d\mu(t) \geq 0
\]

as announced. \( \Box \)

2.2.** Comparing bivariate random vectors**

2.2.1. Bivariate integral stochastic orderings

Numerous stochastic orderings among random vectors have been proposed in the literature. Among these, a remarkable family is the class of the integral stochastic orderings generated by cones of multivariate functions (see, e.g., Marshall (1991)), the natural extension of the univariate orderings (2.1). Specifically, let \( \mathcal{F} \) be a class of measurable functions \( \phi : S \to \mathbb{R} \), where \( S \) is a subset of the real plane \( \mathbb{R}^2 \), and let \( X \) and \( Y \) be a pair of bivariate random vectors valued in \( S \). Then, \( X \) is said to be smaller than \( Y \) for the integral stochastic ordering \( \leq_S \) generated by \( \mathcal{F} \) when

\[
E\phi(X) \leq E\phi(Y) \quad \text{for all functions } \phi \in \mathcal{F}
\]

for which the expectations exist.

As particular cases of special interest, we mention the orderings where \( \mathcal{F} \) is the class of non-decreasing functions \( \mathbb{R}^2 \to \mathbb{R} \), of (non-decreasing) convex and concave functions and of Schur-convex functions (see, e.g., Marshall (1991) and the references therein).

2.2.2. Generating cone of bivariate \( z \)-convex functions

We now introduce several remarkable classes of bivariate real functions whose partial derivatives possess some sign properties. Let \( S \subseteq \mathbb{R}^2 \) and let \( z \in \mathbb{N}_2 \) with \( s_1 + s_2 \geq 1 \). The class \( U_{z_{\text{cx}}}^S \) is defined as

\[
U_{z_{\text{cx}}}^S = \left\{ \phi : S \to \mathbb{R} \left| \frac{\partial^{s_1+s_2} \phi}{\partial x_1^{s_1} \partial x_2^{s_2}} \geq 0 \text{ on } S \right. \right\}.
\]

By analogy with the univariate case and for other reasons given later, the functions in \( U_{z_{\text{cx}}}^S \) are called regular \( z \)-convex functions. The dual class \( U_{z_{\text{cv}}}^S \), the class of the regular \( z \)-concave functions, is defined as

\[
U_{z_{\text{cv}}}^S = \left\{ \phi : S \to \mathbb{R} \left| (-1)^{s_1+s_2+1} \frac{\partial^{s_1+s_2} \phi}{\partial x_1^{s_1} \partial x_2^{s_2}} \geq 0 \text{ on } S \right. \right\}.
\]
It is easily seen that
\[ \phi \in U_{2,icx}^S \iff \phi \in U_{2,cx}^S \] if \( s_1 + s_2 \) is odd,
\[ \phi \in U_{2,icx}^S \iff -\phi \in U_{2,cx}^S \] if \( s_1 + s_2 \) is even. \hfill (2.16)

Let \( K(\xi) = \{ k \in \mathbb{N} | 0 \leq k \leq \xi, k_1 + k_2 \geq 1 \} \). Two more restrictive classes of real functions are the class \( U_{2,icx}^S \) of the regular \( \xi \)-increasing convex functions defined by
\[
U_{2,icx}^S = \bigcap_{k \in K(\xi)} U_{2,cx}^S = \left\{ \phi : S \to \mathbb{R} \mid \frac{\partial^{k_1+k_2} \phi}{\partial x_1^{k_1} \partial x_2^{k_2}} \geq 0 \text{ on } S \text{ for } 0 \leq k \leq \xi, k_1 + k_2 \geq 1 \right\}, \hfill (2.17)
\]
and the class \( U_{2,icv}^S \) of the regular \( \xi \)-increasing concave functions given by
\[
U_{2,icv}^S = \bigcap_{k \in K(\xi)} U_{2,icv}^S = \left\{ \phi : S \to \mathbb{R} \mid (-1)^{k_1+k_2+1} \frac{\partial^{k_1+k_2} \phi}{\partial x_1^{k_1} \partial x_2^{k_2}} \geq 0 \text{ on } S \text{ for } 0 \leq k \leq \xi, k_1 + k_2 \geq 1 \right\}. \hfill (2.18)
\]

Here again, the classes \((2.17)\) and \((2.18)\) are dual in the sense that
\[ \phi(\cdot, \cdot) \in U_{2,icv}^{[b, b]} \iff -\phi(b_1 - \cdot, b_2 - \cdot) \in U_{2,icx}^{[0, b-g]} \], \hfill (2.19)
and
\[ \phi(\cdot, \cdot) \in U_{2,icx}^{R^2} \iff -\phi(-\cdot, -\cdot) \in U_{2,icv}^{R^2}. \hfill (2.20) \]

The functions in \( U_{2,icx}^S \) and \( U_{2,icv}^S \) have an economic meaning as utility functions expressing pairwise risk aversion. Moreover, when \( s_1 = s_2 \), the elements of \( U_{2,icx}^S \) and \( U_{2,icv}^S \) can be characterized in terms of preference among lotteries. For further details on this question, we refer the reader to Scarsini (1985, 1988a) and Mosler and Scarsini (1991), for example.

2.2.3. Bivariate integral orderings of convex-type

The integral stochastic orderings under interest in this paper are generated through \((2.13)\) with the classes \((2.14)-(2.15)\) and \((2.17)-(2.18)\). More precisely, consider two bivariate random vectors \( X \) and \( Y \) valued in \( S \subseteq \mathbb{R}^2 \), \( S \) being the common support of \( X \) and \( Y \) or the “combined support”, i.e. the union of their respective supports. Then, \( X \) is said to be smaller than \( Y \) in the \( \xi \)-convex (resp. \( \xi \)-concave, \( \xi \)-increasing convex and \( \xi \)-increasing concave) ordering, denoted by \( X \preceq_{2,cx} X \) (resp. \( X \preceq_{2,icx} X \), \( X \preceq_{2,icx} Y \) and \( X \preceq_{2,icv} Y \)), when \((2.13)\) holds with \( \mathcal{F} = U_{2,cx}^S \) (resp. \( U_{2,icx}^S \), \( U_{2,icx}^S \) and \( U_{2,icv}^S \)).

We indicate that some special cases of these orderings have been considered before, especially in the economical literature (see, e.g. Bergmann (1978), Atkinson and Bourguignon (1982) and Bergmann (1991), the review in the introduction of Scarsini (1988b) and Mosler and Scarsini (1991), p. 274).

From \((2.17)\) and \((2.18)\), we observe that when \( \xi \leq t \),
\[ U_{2,icx}^S \subset U_{2,icx}^{S,2} \quad \text{and} \quad U_{2,icx}^S \subset U_{2,icx}^{S,1} \]
so that
\[ X \preceq_{2,icx} Y \Rightarrow X \preceq_{2,icx} Y \quad \text{and} \quad X \preceq_{2,icv} Y \Rightarrow X \preceq_{2,icv} Y. \hfill (2.21) \]
In other words, the classes \( \{ \preceq_{2,icx}^S, \xi \in \mathbb{N}_2 \} \) and \( \{ \preceq_{2,icv}^S, \xi \in \mathbb{N}_2 \} \) are hierarchical. Such a property does not hold for \( \preceq_{2,icx}^S \) and \( \preceq_{2,icv}^S \).
The following proposition shows that, without loss of generality, the study may be focused on the single convex case.

**Proposition 2.3.** Let $X$ and $Y$ be random vectors valued in $S$. Then,

$$X \preceq_{S^c} Y \Leftrightarrow X \preceq_{S^c} S^c Y \text{ if } s_1 + s_2 \text{ is odd}, \quad X \preceq_{S^c} Y \preceq_{S^c} S^c X \text{ if } s_1 + s_2 \text{ is even.} \quad (2.22)$$

Moreover,

$$X \preceq_{S^c} (a, b) Y \Leftrightarrow b - Y \preceq_{S^c} (0, b - a) b - X, \quad (2.23)$$

and

$$X \preceq_{S^c} R^2 Y \Leftrightarrow -Y \preceq_{S^c} R^2 X. \quad (2.24)$$

**Proof.** Eq. (2.22) is a direct consequence of (2.16). Now, (2.23) is deduced from (2.19) since

$$X \preceq_{S^c} (a, b) \Leftrightarrow E\phi(X) \leq E\phi(Y) \quad \forall \phi \in \mathcal{U}_{(a, b)}, \quad \mathcal{E}((a, b) \phi)(X) \leq \mathcal{E}(\phi(Y)) \quad \forall \phi \in \mathcal{U}_{(0, b - a)},$$

where the operator $\tau_2$ is defined for a function $\phi : [a, b] \to \mathbb{R}$ by

$$\tau_2 \phi : [0, b - a] \to \mathbb{R}; x \mapsto \phi(b_1 - x, b_2 - x_2);$$

and $\tau_2$ is a bijection between $\mathcal{U}_{(a, b)}$ and $\mathcal{U}_{(0, b - a)}$. Then,

$$X \preceq_{S^c} (a, b) \Leftrightarrow E\phi(b - Y) \leq E\phi(b - X) \quad \forall \phi \in \mathcal{U}_{(0, b - a)} \Leftrightarrow b - Y \preceq_{S^c} (0, b - a) b - X.$$ Finaly, (2.24) follows analogously from (2.10).

In the remainder of the paper, we will concentrate our attention on the ordering $\preceq_{S^c}$. Moreover, the support $S$ will be considered to be of the form $[a, b]$ or $[a, +\infty[$. In this case, one may safely assume that the test functions in $\mathcal{U}_{S^c}$ are non-negative (simply by substituting $\phi(., .) - \phi(a_1, a_2)$ for $\phi(., .)$).

3. Characterizations of $\preceq_{S^c}$

3.1. Starting point

Let us first recall some results on univariate $s$-orderings that are derived in Denuit et al. (1998). Characterizing integral stochastic orderings generally consists in substituting for the generating cone of function $F$ in (2.1) either a dense subclass contained in $F$, or a larger cone corresponding to the closure of $F$ in some suitable topology. The smallest and largest such classes of functions that can be used instead of $F$ are called the minimal and the maximal generators (Müller (1997)). For the $s$-increasing convex ordering among random variables valued in $S = [a, b]$ or $[a, +\infty[$, the minimal generator is the class $\mathcal{U}_{S^c}$ given by

$$\mathcal{U}_{S^c}^S = \{(x - a)^i, i = 1, \ldots, s - 1; (x - t)^{s-1}, t \in \mathbb{R}\}, \quad (3.1)$$

while the maximal generator is the class $\mathcal{U}_{S^c}$ of the $s$-increasing convex functions on $S$ (in the sense of Popoviciu (1933)), defined by

$$\mathcal{U}_{S^c}^S = \{\phi : S \to \mathbb{R}|[x_0, \ldots, x_k] \phi \geq 0 \text{ for all } x_0 \neq \cdots \neq x_k \in S, k = 1, \ldots, s\}. \quad (3.2)$$
In (3.2), [⋯] denotes the classical divided difference operator which is defined recursively, starting from \([x_i] \phi = \phi(x_i)\) for \(i = 0, 1, \ldots, s\), by

\[
[x_0, \ldots, x_s] \phi = \frac{[x_1, \ldots, x_s] \phi - [x_0, \ldots, x_{s-1}] \phi}{x_s - x_0}; \tag{3.3}
\]

(see, e.g., Popoviciu (1940), or Pečarić et al. (1992)). The 1-increasing convex functions on \(S\) are the non-decreasing functions, while the 2-increasing convex functions on \(S\) are simultaneously non-decreasing and convex on \(S\).

Hereafter, we are going to obtain the minimal and maximal generators for the bivariate \(s\)-increasing convex orderings; these will be denoted by \(\mathcal{U}_{s, \text{icx}}^{\mathcal{S}}\) and \(\mathcal{U}_{s, \text{incx}}^{\mathcal{S}}\), respectively.

### 3.2. Minimal generator

Consider for \(s \geq 1\) the class of functions

\[
\mathcal{U}_{s, \text{icx}}^{\mathcal{S}} = \{(x_1 - a_1)^{i_1} (x_2 - a_2)^{i_2}, 0 \leq i_1 \leq s - 1; \ (x_1 - a_1)^{i_1} (x_2 - t_2)^{i_2 - 1}, 0 \leq i_1 \leq s_1 - 1, \ t_2 \in \mathcal{S}_2; \ (x_1 - t_1)^{i_1 - 1} (x_2 - a_2)^{i_2}, 0 \leq i_2 \leq s_2 - 1, \ t_1 \in \mathcal{S}_1; \ (x_1 - t_1)^{i_1 - 1} (x_2 - t_2)^{i_2 - 1}, t \in \mathcal{S}\},
\]

i.e., the functions in \(\mathcal{U}_{s, \text{icx}}^{\mathcal{S}}\) correspond to products of functions in \(\mathcal{U}_{s_1, \text{icx}}^{\mathcal{S}_1}\) and \(\mathcal{U}_{s_2, \text{icx}}^{\mathcal{S}_2}\).

**Characterization 3.1.** Let \(X\) and \(Y\) be two random vectors valued in \(S = [a, b]\) or \([a, +\infty]\). Then, for \(s \geq 1\), \(X \prec S_{s, \text{icx}} Y\) if and only if, (2.13) holds with \(F = \mathcal{U}_{s, \text{icx}}^{\mathcal{S}}\).

**Proof.** Let \(\phi \in \mathcal{U}_{s, \text{icx}}^{\mathcal{S}}\). By Taylor’s expansion of \(\phi\) viewed as a function of \(x_1\) around \(a_1\) (for fixed \(x_2\)), we get

\[
\phi(x_1, x_2) = \sum_{i_1=0}^{s_1-1} \frac{\partial^{i_1} \phi(a_1, x_2) (x_1 - a_1)^{i_1}}{i_1!} + \int_{t_1=a_1}^{x_1} (x_1 - t_1)^{s_1-1} \frac{\partial^{s_1} \phi(t_1, x_2)}{s_1!} \, dt_1. \tag{3.4}
\]

Then, inserting

\[
\frac{\partial^{i_1} \phi(a_1, x_2)}{\partial x_1^{i_1}} = \sum_{i_2=0}^{s_2} \frac{\partial^{i_1+i_2} \phi(a_1, x_2) (x_2 - a_2)^{i_2}}{i_2!} + \int_{t_2=a_2}^{x_2} \frac{(x_2 - t_2)^{s_2-1} \partial^{s_2} \phi(t_1, x_2)}{(s_2-1)!} \, dt_2,
\]

and

\[
\frac{\partial^{s_1} \phi(t_1, x_2)}{\partial x_1^{s_1}} = \sum_{i_2=0}^{s_2} \frac{\partial^{s_1+i_2} \phi(t_1, x_2) (x_2 - a_2)^{i_2}}{i_2!} + \int_{t_2=a_2}^{x_2} \frac{(x_2 - t_2)^{s_2-1} \partial^{s_2} \phi(t_1, x_2)}{(s_2-1)!} \, dt_2,
\]

in (3.4) and using Fubini’s theorem, we obtain that
\[ E \phi(X) = \sum_{i_1=0}^{s_1-1} \sum_{i_2=0}^{s_2-1} \frac{\partial^{i_1+i_2} \phi(a_1, a_2) E[(X_1 - a_1)^{i_1}(X_2 - a_2)^{i_2}]}{\partial x_1^{i_1} \partial x_2^{i_2}} t_1! t_2! \]

\[ + \sum_{i_1=0}^{s_1-1} \int_{t_2=a_2}^{+\infty} \frac{E[(X_2 - t_2)^{s_2-1}(X_1 - a_1)^{i_1}]}{(s_2-1)!} i_1! \frac{\partial^{i_1+i_2} \phi(a_1, t_2)}{\partial x_1^{i_1} \partial x_2^{i_2}} dt_2 \]

\[ + \sum_{i_2=0}^{s_2-1} \int_{t_1=a_1}^{+\infty} \frac{E[(X_1 - t_1)^{s_1-1}(X_2 - a_2)^{i_2}]}{(s_1-1)!} i_2! \frac{\partial^{i_1+i_2} \phi(t_1, a_2)}{\partial x_1^{i_1} \partial x_2^{i_2}} dt_1 \]

\[ + \int_{t_1=a_1}^{+\infty} \int_{t_2=a_2}^{+\infty} \frac{E[(X_1 - t_1)^{s_1-1}(X_2 - t_2)^{s_2-1}]}{(s_1-1)! (s_2-1)!} \frac{\partial^{i_1+i_2} \phi(t_1, t_2)}{\partial x_1^{i_1} \partial x_2^{i_2}} dt_1 dt_2, \tag{3.5} \]

which achieves the proof of the “\( \leq \)”-part. The “\( \Rightarrow \)”-part is obvious since the functions in \( \mathcal{U}_{\text{icx}}^S \) can be obtained as a uniform limit of sequences of functions in \( \mathcal{U}_{\text{icx}} \) (see, e.g., Denuit et al. (1998), proof of Theorem 3.5).

For \( \xi = 1 \) and \( \bar{\xi} \), the above result becomes very simple.

**Corollary 3.2.** Let \( X \) and \( Y \) be random vectors valued in \([a, b]\) or \([a, +\infty[.\) Then, \( X \preceq_{\text{icx}} Y \) if, and only if,

\[ P[X_1 > t_1, X_2 > t_2] \leq P[Y_1 > t_1, Y_2 > t_2], \quad t \in S, \]

while \( X \preceq_{\text{icx}} Y \) if, and only if,

\[ \left\{ \begin{array}{l}
E[(X_i - t_i)_+] \leq E[(Y_i - t_i)_+], \quad t_i \in S_i \quad \text{with } i = 1, 2, \\
E[(X_1 - t_1)_+(X_2 - t_2)_+] \leq E[(Y_1 - t_1)_+(Y_2 - t_2)_+], \quad t \in S.
\end{array} \right. \]

The second part of Corollary 3.2 provides an actuarial interpretation of the ordering \( \preceq_{\text{icx}}^S \); the bivariate risks \( X \) and \( Y \) satisfy \( X \preceq_{\text{icx}}^S Y \) if, and only if, their univariate components are ordered in the stop-loss sense and the right-tail of \( X \) is “less dangerous” than that of \( Y \).

### 3.3. Iterated right-tail distributions

The \( \xi \)-increasing convex orderings can also be characterized in terms of iterated right-tail distributions. Given a random vector \( X \), its \( \xi \)-iterated right-tail distributions, denoted by \( S_{\gamma}(X; \gamma) \) for \( \gamma \in \mathbb{N}_2 \) such that \( s_1 + s_2 \geq 1 \) and \( x \in \mathbb{R}_2 \), are defined recursively by

\[ S_{\gamma}(X; \gamma) = \int_{t_1=x_1}^{+\infty} S_{(s_1, s_2-1)}(X; (t_1, x_2)) dt_1 \]

\[ = \int_{t_2=x_2}^{+\infty} S_{(s_1-1, s_2)}(X; (x_1, t_2)) dt_2, \tag{3.6} \]

with \( S_{(1,0)}(X; \gamma) = P[X_1 > x_1], \ S_{(0,1)}(X; \gamma) = P[X_2 > x_2], \ S_1(X; \gamma) = P[X_1 > x_1, X_2 > x_2]. \) It is known (see, e.g., Denuit et al. (1998), proof of Theorem 3.3) that

\[ S_{(s_1,0)}(X; \gamma) = \frac{E[(X_1 - x_1)^{s_1-1}]}{(s_1-1)!} \quad \text{and} \quad S_{(0,s_2)}(X; \gamma) = \frac{E[(X_2 - x_2)^{s_2-1}]}{(s_2-1)!}. \]

We now derive an explicit expression for \( S_{\gamma}(X; \cdot) \) when \( \gamma \geq 1 \).

**Lemma 3.3.** For any \( \gamma \geq 1 \) and \( x \in \mathbb{R}_2 \),

\[ S_{\gamma}(X; \gamma) = \frac{1}{(s_1-1)!(s_2-1)!} E[(X_1 - x_1)^{s_1-1}(X_2 - x_2)^{s_2-1}]. \]
Proof. We proceed by induction. The result is obvious for \( s = 1 \). Then, by the recurrence assumption and using Fubini’s theorem, we obtain

\[
S_{(s_1+1,s_2)}(X; \mathbf{x}) = \int_{t_1=x_1}^{+\infty} \frac{E[(X_1 - t_1)^{s_1-1}(X_2 - x_2)^{s_2-1}]}{(s_1-1)!(s_2-1)!} \, dt_1
\]

\[
= \frac{1}{(s_1-1)!(s_2-1)!} E \left[ (X_2 - x_2)^{s_2-1} \int_{t_1=x_1}^{+\infty} (X_1 - t_1)^{s_1-1} \, dt_1 \right]
\]

\[
= \frac{1}{s_1!(s_2-1)!} E[(X_1 - x_1)^{s_1}(X_2 - x_2)^{s_2-1}],
\]

and similarly for \( S_{(s_1,s_2+1)}(X; \mathbf{x}) \).

Combining Characterization 3.1 and Lemma 3.3, we get the next result.

**Characterization 3.4.** Let \( X \) and \( Y \) be random vectors valued in \([a, b]\) or \([a, +\infty)\). Then, for \( s \geq 1 \),

\[
X \preceq^S_{\mathbf{S}_k} Y
\]

if, and only if,

\[
\begin{align*}
S_k(X; a) &\leq S_k(Y; a), \quad 0 \leq k \leq s, \quad k_1 + k_2 \geq 1, \\
S_{(s_1,s_2)}(X; (t_1, a_2)) &\leq S_{(s_1,s_2)}(Y; (t_1, a_2)), \quad 0 \leq s_2 \leq s, \quad t_1 \in \mathcal{S}_1, \\
S_{(s_1,s_2)}(X; (a_1, t_2)) &\leq S_{(s_1,s_2)}(Y; (a_1, t_2)), \quad 0 \leq k_1 \leq s_1, \quad t_2 \in \mathcal{S}_2, \\
S_k(X; t) &\leq S_k(Y; t), \quad t \in \mathcal{S}.
\end{align*}
\]

3.4. **Maximal generator**

3.4.1. **Bivariate divided differences**

To begin with, we introduce a bivariate divided difference operator. We mention that several bidimensional extensions of the operator (3.3) have been proposed by Popoviciu (1933). We opt here for the so-called **partial divided difference** operator. It is defined by reference to a grid of points corresponding to the intersections of \( k_1 + 1 \) parallels to \( Ox \) and \( k_2 + 1 \) parallels to \( Oy \) (such a grid will be subsequently referred to as a \( k \)-th degree grid). Let \( \phi \) be a real-valued function defined on \( \mathcal{S} \). Then, the \( k \)-th degree divided difference of \( \phi \) at distinct points \( x_0, \ldots, x_{k_1} \) in \( \mathcal{S}_1 \) and \( y_0, \ldots, y_{k_2} \) in \( \mathcal{S}_2 \) is defined by

\[
\phi = \begin{bmatrix} x_0, \ldots, x_{k_1} \\ y_0, \ldots, y_{k_2} \end{bmatrix} \phi = [x_0, \ldots, x_{k_1}]([y_0, \ldots, y_{k_2}] \phi) = [y_0, \ldots, y_{k_2}]([x_0, \ldots, x_{k_1}] \phi).
\]

(3.7)

It is worth recalling some basic properties of this operator (see Popoviciu (1933, 1940, 1942)).

3.4.2. **Expansion in terms of the values of \( \phi \)**

Eq. (3.7) can be expressed as a linear combination of the values of \( \phi \) computed at the points \( ((x_i, y_j), 0 \leq i_1 \leq k_1, 0 \leq i_2 \leq k_2) \). Specifically,

\[
\phi = \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2} \frac{\phi(x_{i_1}, y_{i_2})}{c_1(x_{i_1})c_2(y_{i_2})},
\]

(3.8)
where
\[
\omega_1(x_i) = \prod_{j_1=0; j_1 \neq i_1}^{k_1} (x_i - x_{j_1}), \quad \text{and} \quad \omega_2(y_i) = \prod_{j_2=0; j_2 \neq i_2}^{k_2} (y_i - y_{j_2}).
\]

As a consequence, (3.7) is a linear operator and commutes with the limit.

3.4.3. Determinantal form
Eq. (3.7) can also be written as the ratio of two determinants. Let
\[(3.9)\]
\[
\begin{bmatrix} x_0, \ldots, x_{k_1} \\ y_0, \ldots, y_{k_2} \end{bmatrix} \Phi = \frac{\delta_\Phi \left( x_0, \ldots, x_{k_1} ; y_0, \ldots, y_{k_2} ; \phi \right)}{\delta_\Phi \left( x_0, \ldots, x_{k_1} ; x_{k_1}^1, x_{k_2}^2 ; \phi \right)}.
\]

3.4.4. Bivariate $\varepsilon$-convex functions
A function $\phi : \mathcal{S} \to \mathbb{R}$ is said to be $\varepsilon$-convex if
\[(3.11)\]
\[
\begin{bmatrix} x_0, \ldots, x_{s_1} \\ y_0, \ldots, y_{s_2} \end{bmatrix} \phi \geq 0 \quad \text{for all distinct points} \quad x_0, \ldots, x_{s_1} \in \mathcal{S}_1, \ y_0, \ldots, y_{s_2} \in \mathcal{S}_2.
\]

The class of all the functions $\phi : \mathcal{S} \to \mathbb{R}$ satisfying (3.11) is denoted by $U_{\varepsilon-\text{cx}}^S$. Similar to (2.17), $U_{\varepsilon-\text{icx}}^S$, the class of the bivariate $\varepsilon$-increasing convex functions, is defined by $U_{\varepsilon-\text{icx}}^S = \bigcap_{k \in K(\varepsilon)} U_{\varepsilon-k}^S$.

3.4.5. Partial derivatives
If the partial derivative of $\phi$ of degree $s_1 + s_2$ exists, then
\[(3.12)\]
\[
\phi \in U_{\varepsilon-\text{cx}}^S \Leftrightarrow \frac{\partial^{s_1 + s_2} \phi}{\partial x_1^{s_1} \partial x_2^{s_2}} \geq 0 \quad \text{on the domain} \quad \mathcal{S}.
\]

so that $U_{\varepsilon-\text{cx}}^S \subset U_{\varepsilon-\text{icx}}^S$ and $U_{\varepsilon-\text{icx}}^S \subset U_{\varepsilon-\text{cx}}^S$. In general, not all the functions in $U_{\varepsilon-\text{icx}}^S$ have a partial derivative of degree $s_1 + s_2$. Nevertheless, the $\varepsilon$-convexity implies certain regularity conditions. So, a function $\phi \in U_{\varepsilon-\text{icx}}^S$ has a partial derivative
\[(3.13)\]
\[
\frac{\partial^{r_1 + r_2} \phi}{\partial x_1^{r_1} \partial x_2^{r_2}} \quad \text{for any} \quad r \quad \text{such that} \quad \frac{r_1}{s_1} + \frac{r_2}{s_2} < 1.
\]

Moreover, if $\phi \in U_{\varepsilon-\text{icx}}^S$ and if the partial derivative of degree $r_1 + r_2$ exists, then
\[(3.14)\]
\[
\frac{\partial^{r_1 + r_2} \phi}{\partial x_1^{r_1} \partial x_2^{r_2}} \in U_{(\varepsilon-\text{icx})}^S.
\]
3.4.6. Product invariance

Let \( \mathcal{U}_{S_1}^{S_1,1+} \) and \( \mathcal{U}_{S_2}^{S_2,1+} \) denote the restriction of \( \mathcal{U}_{S_1}^{S_1} \) and \( \mathcal{U}_{S_2}^{S_2} \) to their non-negative elements. It is directly seen that

\[
\phi_1 \in \mathcal{U}_{S_1}^{S_1,1+} \quad \text{and} \quad \phi_2 \in \mathcal{U}_{S_2}^{S_2,1+} \Rightarrow \phi_1 \cdot \phi_2 \in \mathcal{U}_{S}^{S}\tag{3.15}
\]

We are now in a position to prove that the maximal generator of the ordering \( \preceq_{S_1,1} \) is the class \( \mathcal{U}_S \).

Characterization 3.5. Let \( X \) and \( Y \) be two random vectors valued in \( S = [a, b] \) or \([a, +\infty[\). Then, \( X \preceq_{S_1,1} Y \) if, and only if, (2.13) is satisfied with \( \mathcal{F} = \mathcal{U}_S^{S} \).

Proof. The “\( \Rightarrow \)”-part is obvious since \( \mathcal{U}_S^{S} \subset \mathcal{U}_S^{S} \). To get the converse, it can be shown that every \( \phi \in \mathcal{U}_S^{S} \) is the uniform limit of a sequence of functions \( \{\phi_n, n \in \mathbb{N}\} \) that can be expressed as linear combinations (with non-negative coefficients) of functions in \( \mathcal{U}_S^{S} \). Therefore, \( E\phi_n(X) \to E\phi(X) \) and \( E\phi_n(Y) \to E\phi(Y) \) as \( n \to +\infty \), and by hypothesis \( E\phi_n(X) \leq E\phi_n(Y) \) for all \( n \), which yields \( E\phi(X) \leq E\phi(Y) \). \( \square \)

3.5. Product orderings

Recently, Denuit and Lefèvre (1997b) introduced a bidimensional extension a priori different, of the \( s \)-increasing convex orderings that is of product-type. More precisely, \( X \) is said to be smaller than \( Y \) in the \( s_1 \times s_2 \)-increasing convex sense, denoted by \( X \preceq_{S_1 \times S_2} Y \), if (2.13) holds with for \( \mathcal{F} \) the class \( \mathcal{U}_S^{S} \).

The proposition below states that this ordering is in fact equivalent to the ordering investigated here.

Proposition 3.6. Let \( X \) and \( Y \) be random vectors valued in \( S \). Then

\[ X \preceq_{S_1 \times S_2} Y \Leftrightarrow X \preceq_{S_1 \times S_2} Y. \]

Proof. The “\( \Rightarrow \)”-part is obvious since \( \mathcal{U}_S^{S} \subset \mathcal{U}_S^{S} \). The “\( \Leftarrow \)”-part follows from Characterization 3.4 (because the functions \( x \mapsto (x-a)^i, i = 1, \ldots, s-1 \), and \( x \mapsto (x-t)^s \) (for any fixed \( t \in \mathbb{R} \)) all belong to \( \mathcal{U}_S^{S} \) (see, e.g., Denuit et al. (1998), proof of Theorem 3.2). \( \square \)

By Property 2.1 in Denuit and Lefèvre (1997b), when the random vectors \( X \) and \( Y \) possess the same marginal distributions, then

\[ X \preceq_{S_1 \times S_2} Y \Leftrightarrow (2.13) \text{ holds with } \mathcal{F} = \mathcal{U}_S^{S_1} \times \mathcal{U}_S^{S_2}, \tag{3.17} \]

i.e., the assumption of non-negativity of the test functions may be dropped in this case. Moreover, the other results derived in Denuit and Lefèvre (1997b) are thus applicable here too. Note that the \( 1 \)-increasing convex ordering is the classical upper orthant order, while the \( 2 \)-increasing convex ordering is the orthant convex order.
4. Properties of \( \preceq_{\text{2-icx}} \)

4.1. Closure properties

**Property 4.1.** Let \( X \) and \( Y \) be two random vectors valued in \( S \). Consider the subsets \( T_1 \) and \( T_2 \) of \( \mathbb{R} \) and the functions \( \psi_1 : S_1 \to T_1 \in \mathcal{U}_{1-icx}^S \) and \( \psi_2 : S_2 \to T_2 \in \mathcal{U}_{2-icx}^S \). Then

\[
X \preceq_{\text{2-icx}} Y \Rightarrow [\psi_1(X_1), \psi_2(X_2)] \preceq_{\text{2-icx}} [\psi_1(Y_1), \psi_2(Y_2)].
\]

**Proof.** As shown by Popoviciu (1933), p. 26, the composition of two \( s \)-increasing convex functions is still a \( s \)-increasing convex function, so that the result follows from Proposition 3.6.

As a consequence, we get that \( \preceq_{\text{2-icx}} \) is scale and shift invariant, i.e., for all \( \alpha_1, \alpha_2 \in \mathbb{R}^+, \beta_1, \beta_2 \in \mathbb{R} \):

\[
X \preceq_{\text{2-icx}} Y \Rightarrow (\alpha_1X_1 + \beta_1, \alpha_2X_2 + \beta_2) \preceq_{\text{2-icx}} (\alpha_1Y_1 + \beta_1, \alpha_2Y_2 + \beta_2), \tag{4.1}
\]

where \( T = \{ y \in \mathbb{R}^2 \}^\times = (\alpha_1X_1 + \beta_1, \alpha_2X_2 + \beta_2) \) for some \( x \in S \).

For simplicity, all the random vectors hereafter are assumed to be valued in \( \mathbb{R}^+_2 \). The results, however, can be rewritten in whole generality.

**Property 4.2.** Let \( X \) and \( Y \) be random vectors valued in \( \mathbb{R}^+_2 \). If \( \Phi \) is a measurable function \( \mathbb{R}^+_2 \times \mathbb{R}_2 \to \mathbb{R}^+_2 \) and \( \Theta \) is a random vector valued in a subset \( T \) of \( \mathbb{R}^+_2 \), independent from \( X \) and \( Y \), then

\[
\Phi(X, \Theta) \preceq_{\text{2-icx}} \Phi(Y, \Theta) \quad \text{for all } \Theta \in T \Rightarrow \Phi(X, \Theta) \preceq_{\text{2-icx}} \Phi(Y, \Theta).
\]

**Proof.** The proof is immediate, since for any \( \phi \in \mathcal{U}_{\text{2-icx}}^T \),

\[
E[\Phi(X, \Theta)] = \int_{\Theta} \int_{T_1} \int_{T_2} E[\Phi(X, \Theta)] dP(\Theta_1 \leq \theta_1, \Theta_2 \leq \theta_2) \\
\leq \int_{\Theta} \int_{T_1} \int_{T_2} E[\Phi(Y, \Theta)] dP(\Theta_1 \leq \theta_1, \Theta_2 \leq \theta_2) = E[\Phi(Y, \Theta)].
\]

In particular, \( \preceq_{\text{2-icx}} \) is closed under mixtures, i.e., under the assumptions of Property 4.2, if \( [X|\Theta = \theta] \preceq_{\text{2-icx}} [Y|\Theta = \theta] \) for all \( \theta \) in the support of \( \Theta \), then \( X \preceq_{\text{2-icx}} Y \).

**Property 4.3.**

(i) \( \preceq_{\text{2-icx}} \) is closed under convolution, i.e. if \( X_1, \ldots, X_k \) (resp. \( Y_1, \ldots, Y_k \)) are \( k \) independent random vectors valued in \( \mathbb{R}^+_2 \), then

\[
X_i \preceq_{\text{2-icx}} Y_i, \quad i = 1, \ldots, k \Rightarrow \sum_{i=1}^k X_i \preceq_{\text{2-icx}} \sum_{i=1}^k Y_i.
\]

(ii) \( \preceq_{\text{2-icx}} \) is closed under compounding, i.e. if \( \{X_i, i \geq 1\} \) and \( \{Y_j, j \geq 1\} \) are two sequences of independent random vectors valued in \( \mathbb{R}^+_2 \), and if \( N \) is a non-negative integer-valued random variable which is independent of the \( X_i \)'s and of the \( Y_j \)'s, then

\[
X_i \preceq_{\text{2-icx}} Y_i, \quad i \geq 1 \Rightarrow \sum_{i=1}^N X_i \preceq_{\text{2-icx}} \sum_{i=1}^N Y_i.
\]
**Proof.** For (i), we first observe from (4.1) that for any $z \in \mathbb{R}^+_2$,

$$X \preceq_{2-icx} Y \Rightarrow X + z \preceq_{2-icx} Y + z.$$ 

Thus, if $Z$ is a random variable independent from $X$ and $Y$, we obtain by Property 4.2 that

$$X \preceq_{2-icx} Y \Rightarrow X + Z \preceq_{2-icx} Y + Z. \quad (4.2)$$

We now proceed by induction with respect to $k$. Note that, without loss of generality, we may assume that the $X_i$'s and the $Y_j$'s are mutually independent (since the orderings $\preceq_{2-icx}$ do not depend on the joint distribution of the random vectors to be compared). Applying (4.2), we then obtain from the recurrence hypothesis that

$$\sum_{i=1}^{k+1} X_i \preceq_{2-icx} \sum_{i=1}^k Y_i + X_{k+1}. \quad (4.3)$$

On the other hand, as $X_{k+1} \preceq_{2-icx} Y_{k+1}$, we also have from (4.2) that

$$X_{k+1} + \sum_{i=1}^k Y_i \preceq_{2-icx} \sum_{i=1}^{k+1} Y_i. \quad (4.4)$$

Combining (4.3) and (4.4) yields the result. For (ii), it suffices to apply (i) with the closure property under mixture. \hfill $\square$

### 4.2. Univariate versus bivariate orderings

Bivariate $s$-increasing convex orderings imply interesting ordering results on their univariate components.

**Property 4.4.** Let $X$ and $Y$ be random vectors valued in $\mathbb{R}^+_2$.

(i) Then, for any function $\phi : \mathbb{R}^+_2 \to \mathbb{R}^+$, $\mathcal{U}_{2-icx}$,

$$X \preceq_{2-icx} Y \Rightarrow \phi(X) \preceq_{(s_1+s_2)-icx} \phi(Y). \quad (4.5)$$

In particular,

$$X \preceq_{2-icx} Y \Rightarrow \alpha X_1 + \beta X_2 \preceq_{(s_1+s_2)-icx} \alpha Y_1 + \beta Y_2, \quad \forall \alpha, \beta \in \mathbb{R}^+. \quad (4.6)$$

(ii) Assume that $X$ and $Y$ are such that

$$\text{Cov}[\phi_1(X_1), \phi_2(X_2)] \leq \text{Cov}[\phi_1(Y_1), \phi_2(Y_2)] \quad \text{for all } \phi_1 \in \mathcal{U}^{\mathbb{R}^+}_{s_1-icx}, \; \phi_2 \in \mathcal{U}^{\mathbb{R}^+}_{s_2-icx}. \quad (4.7)$$

Then

$$X \preceq_{2-icx} Y \Leftrightarrow X_1 \preceq_{s_1-icx} Y_1 \quad \text{and} \quad X_2 \preceq_{s_2-icx} Y_2. \quad (4.8)$$
Proof. To get (4.5), it suffices to observe that if $\psi \in U_{(s_{1}+s_{2})-\icx}^{R^{+}}$ and $\phi : R^{+} \rightarrow R^{+} \in U_{(s_{2}-\icx)_{+}}^{R^{+}}$, then $\psi \circ \phi \in U_{(s_{2}-\icx)_{+}}^{R^{+}}$. (4.6) follows from (4.5) by taking $\phi(x) = ax_{1} + bx_{2}$. Now, let us prove (ii). The "$\Rightarrow$"-part of (4.8) is a particular case of (4.6). For the "$\Leftarrow$"-part, we introduce the following classes of functions: $R^{+} \rightarrow R^{+}$:

$$V_{1} = \{ x_{1}^{i_{1}}, 0 \leq i_{1} \leq s_{1} - 1; (x_{1} - t_{1})_{+}^{s_{1} - 1}, t_{1} \in R^{+} \}$$

and

$$V_{2} = \{ x_{2}^{i_{2}}, 0 \leq i_{2} \leq s_{2} - 1; (x_{2} - t_{2})_{+}^{s_{2} - 1}, t_{2} \in R^{+} \}.$$ 

Note that $V_{1} \subset U_{s_{1}-\icx}^{R^{+}}$ and $V_{2} \subset U_{s_{2}-\icx}^{R^{+}}$. From the hypothesis, we then obtain that for any $\phi_{1} \in V_{1}$ and $\phi_{2} \in V_{2}$,

$$0 \leq E\phi_{1}(X_{1}) \leq E\phi_{1}(Y_{1}), \quad \text{and} \quad 0 \leq E\phi_{2}(X_{2}) \leq E\phi_{2}(Y_{2}),$$

which yields

$$E\phi_{1}(X_{1})E\phi_{2}(X_{2}) \leq E\phi_{1}(Y_{1})E\phi_{2}(Y_{2}). \quad (4.9)$$

From (4.9) and using the basic inequality (4.7), we deduce that

$$E[\phi_{1}(X_{1})\phi_{2}(X_{2})] \leq E[\phi_{1}(Y_{1})\phi_{2}(Y_{2})] \quad \text{for all } \phi_{1} \in V_{1}, \ \phi_{2} \in V_{2},$$

hence the result by Characterization 3.1. \qed

We mention that Property 4.4 remains valid with $\preceq_{2-\icx}^{R^{+}}$ substituted for $\preceq_{2-\icx}^{R^{+}}$. It is worth emphasizing the implication (4.6) in this case. When $z = 1$ for instance, it says that if $X \preceq_{1-\icx}^{\infty} Y$, then

$$\alpha X_{1} + \beta X_{2} \preceq_{2-\icx}^{R^{+}} \alpha Y_{1} + \beta Y_{2}, \quad \forall \alpha, \beta \in R^{+}. \quad (4.10)$$

A known implication (Baccelli and Makowski (1989), formula (15)) is that then

$$\alpha X_{1} + \beta X_{2} \preceq_{1_{1}} \alpha Y_{1} + \beta Y_{2}. \quad (4.11)$$

By Proposition 2.1, $\preceq_{1_{1}}$ is equivalent to $\preceq_{\infty-\icx}^{R^{+}}$, so that (4.10) provides a reinforcement of (4.11).

The next proposition points out two simple special situations where, starting from ordered random variables, it is possible to build ordered random vectors. The proof is immediate from Property 4.4.

Property 4.5.

(i) Let $X$ and $Y$ be random variables valued in $S_{1}$ such that $X \preceq_{1_{1}}^{S_{1}} Y$, and let $Z$ be a random variable valued in $S_{2}$ and independent from $X$ and $Y$. Then, for any $s_{2} \in N$, $(X, Z) \preceq_{(S_{2}+S_{1})-\icx}^{1_{1}} (Y, Z)$.

(ii) Let $X_{i}$ and $Y_{i}$ be random variables valued in $S_{i}$ such that $X_{i} \preceq_{1_{1}}^{S_{i}} Y_{i}, i = 1, 2$. Then, $X_{1}^{\perp} \preceq_{1_{1}}^{S_{1}} Y_{1}^{\perp}$, where $X^{\perp}$ (resp. $Y^{\perp}$) is the random vector with the same marginal distributions than $X$ (resp. $Y$) but with independent components.

4.3. Orderings of dependence

Let $R(F_{1}, F_{2})$ denote the class of all the random vectors $X$ with fixed marginal distributions, $F_{1}$ for $X_{1}$ and $F_{2}$ for $X_{2}$. By Proposition 3.6 and (3.17), we see that if $X$ and $Y \in R(F_{1}, F_{2})$, then $X \preceq_{2-\icx}^{S_{1}} Y$ is equivalent to

$$\text{Cov}[\phi_{1}(X_{1}), \phi_{2}(X_{2})] \leq \text{Cov}[\phi_{1}(Y_{1}), \phi_{2}(Y_{2})], \quad \forall \ \phi_{1} \in U_{s_{1}-\icx}^{S_{1}}, \ \phi_{2} \in U_{s_{2}-\icx}^{S_{2}}. \quad (4.12)$$
In other words, the structure of dependence expressed in $X$ is smaller than that in $Y$ : $X$ is said to be less $\varepsilon$-correlated than $Y$.

In particular, let us associate with any random vector $X \in \mathcal{R}(F_1, F_2)$ its independent version $X^\perp$. $X$ is said to be negatively $\varepsilon$-dependent if $X$ is less $\varepsilon$-correlated than $X^\perp$, i.e., when $X \preceq S_{\varepsilon-icx} X^\perp$, which is equivalent to

$$E[\phi_1(X_1)\phi_2(X_2)] \leq E\phi_1(X_1)E\phi_2(X_2), \quad \forall \phi_1 \in \overline{U}_{S_1-icx}^S, \quad \phi_2 \in \overline{U}_{S_2-icx}^S. \quad (4.13)$$

Similarly, $X$ is said to be positively $\varepsilon$-dependent when $X^\perp \preceq S_{\varepsilon-icx} X$

Now, note that in $\mathcal{R}(F_1, F_2)$, the assumption (4.7) in Property 4.4 is equivalent to (4.12). From Property 4.4 (i) and (ii), we directly deduce the following result.

**Proposition 4.6.** Let $X$ and $Y$ be random vectors valued in $\mathbb{R}^+_2$ such that $X$ is less $\varepsilon$-correlated than $Y$, with $X_1 \preceq_{S_1-icx} Y_1$, $X_2 \preceq_{S_2-icx} Y_2$. Then

$$\alpha X_1 + \beta X_2 \preceq_{(S_1+S_2)-icx} \alpha Y_1 + \beta Y_2, \quad \forall \alpha, \beta \in \mathbb{R}^+.$$

5. $S_{l-icx}$ ordering in $\mathcal{R}(F_1, F_2)$

**Equivalence with $\preceq_{l-icx}$**. Let $X$ and $Y$ be random vectors in $\mathcal{R}(F_1, F_2)$. By Corollary 3.2,

$$X \preceq_{S_{l-icx}} Y \iff P[X_1 > t_1, X_2 > t_2] \leq P[Y_1 > t_1, Y_2 > t_2] \quad \text{for all } t \in \mathbb{R}_2, \quad (5.1)$$

and since the marginal distributions are identical,

$$X \preceq_{S_{l-icx}} Y \iff P[X_1 \leq t_1, X_2 \leq t_2] \leq P[Y_1 \leq t_1, Y_2 \leq t_2] \quad \text{for all } t \in \mathbb{R}_2. \quad (5.2)$$

It is known (see, e.g., Cambanis et al. (1976), Tchen (1980) and Rüschendorf (1980)) that (5.1) and (5.2) are equivalent to $E\phi(X) \leq E\phi(Y)$ for all the functions $\phi$ satisfying

$$\phi(x_1, y_1) + \phi(x_0, y_0) - \phi(x_1, y_0) - \phi(x_0, y_1) \geq 0 \quad \text{for all } x_0 \leq x_1, \ y_0 \leq y_1 \in \mathbb{R},$$

which can be rewritten as

$$\begin{bmatrix} x_0, & x_1 \\ y_0, & y_1 \end{bmatrix} \phi \geq 0, \quad \text{for all } x_0 \leq x_1, \ y_0 \leq y_1 \in \mathbb{R},$$

i.e., for any function $\phi \in \mathcal{U}_{l-icx}^S$. In other words, $\preceq_{l-icx}$ and $\preceq_{l-icx}^S$ are equivalent in $\mathcal{R}(F_1, F_2)$. The functions $\phi$ in $\mathcal{U}_{l-icx}^S$ are called *quasi-monotone, superadditive or supermodular* (see, e.g., Szekli et al. (1994), Bäuerle (1997) and Shaked and Shanthikumar (1997)). When $X \preceq_{l-icx} Y$, $Y$ is said to be more concordant, or more positively quadrant dependent than $X$, or to stochastically dominate $X$.

**Fréchet bounds.** As shown by Hoeffding (1940) and Fréchet (1951) (see also, e.g., Whitt (1976) and Rüschendorf (1981)), the distribution functions in $\mathcal{R}(F_1, F_2)$ admit a maximum $H^+$ and a minimum $H^-$ given by

$$H^+(x) = \min[F_1(x_1), F_2(x_2)] \quad \text{and} \quad H^-(x) = [F_1(x_1) + F_2(x_2) - 1]^+, \quad x \in \mathbb{R}_2. \quad (5.3)$$

Clearly, $H^+$ and $H^-$ are the distribution functions of the following random vectors

$$X^{(1)}_{\text{max}} = [F_1^{-1}(U), F_2^{-1}(U)] \quad \text{and} \quad X^{(1)}_{\text{min}} = [F_1^{-1}(U), F_2^{-1}(1 - U)].$$
respectively, where \( F_i^{-1} \), \( i = 1, 2 \), represent the quantile functions and \( U \) denotes a random variable uniformly distributed on \([0, 1]\). Thus, \( X^{(1)}_\text{max} \) and \( Y^{(1)}_\text{min} \) correspond to the extrema in \( \mathcal{R}(F_1, F_2) \) with respect to \( \preceq_1 \). In actuarial sciences, these bounds can be used to construct bounds on multilife premiums (see, e.g., Carrière and Chan (1986), Denuit and Lefèvre (1997b) and Denuit et al. (1999a)).

**Lorentz inequality.** Let \( X \in \mathcal{R}(F, F) \), i.e. \( X_1 \) and \( X_2 \) are assumed to be identically distributed (but not necessarily independent). A direct corollary of (5.3) is that

\[
X = (X_1, X_2) \preceq_1 \text{L-cx} (X_1, X_1).
\]

(5.4)

This corresponds to a standard inequality sometimes referred to as the Lorentz inequality (see, e.g., Bäuerle (1997)).

In particular, combining (5.4) and (4.6), we get that

\[
\alpha X_1 + \beta X_2 \preceq_2 \text{L-cx} (\alpha + \beta) X_1 \quad \forall \alpha, \beta \in \mathbb{R}^+.
\]

(5.5)

The comparison (5.5) has been used in actuarial sciences to measure the impact of some dependence between the risks of a portfolio on the stop-loss premiums (see, e.g., Heilmann (1986), Kaas and Hesselager (1995), Exercise 2, Note 2, and Dhaene and Goovaerts (1996, 1997), Corollary 1).

**Correlation order.** The ordering \( \preceq_2 \text{L-cx} \) in \( \mathcal{R}(F_1, F_2) \) has been considered by Dhaene and Goovaerts (1996,1997) under the name of correlation order. These authors wanted to find an ordering between \( X \) and \( Y \) such that the sum of their components are ordered in the stop-loss sense. They proved that the correlation order satisfies this condition, a result which can be viewed as a special case of Property 4.4 (i) with \( s_1 = s_2 = 1 \).

**Quadrant dependence.** The classical notions of positive and negative quadrant dependence (\( PQD \) and \( NQD \), in short) are defined as follows: a bivariate random vector \( X \) is said to be \( PQD \) (resp. \( NQD \)) if

\[
\text{Cov}[\phi_1(X_1), \phi_2(X_2)] \geq 0 \quad \text{(resp.} \leq 0) \quad \text{for all non-decreasing functions} \quad \phi_1, \phi_2.
\]

(5.6)

In our framework, \( PQD \) and \( NQD \) thus correspond to the positive and negative \( 1 \)-dependence, respectively. See, e.g., Szekli (1995), as well as Norberg (1989) for some applications in life insurance tariffication.

The result below is quite immediate.

**Proposition 5.1.** Let \( X \) and \( Y \) be random vectors valued in \( \mathbb{R}^2_+ \) such that \( X \) is \( NQD \) and \( Y \) is \( PQD \). Then, the equivalence (4.8) holds for \( s = 1 \), and

\[
X \preceq_2 \text{L-cx} Y \quad \text{and} \quad Y \preceq_2 \text{L-cx} X.
\]

(5.7)

From Property 4.3, we get the following stability properties of the \( PQD \).

**Property 5.2.**

(i) If \( X \) is a \( PQD \) random vector independent of the vector \( Z \), then \( X + Z \) is \( PQD \).

(ii) If \( X_1, X_2, \ldots, X_n \) are \( PQD \) independent random vectors, then \( \sum_{i=1}^n X_i \) is \( PQD \).

(iii) If, moreover, \( N \) denotes a non-negative integer-valued random variable, independent of the \( X_i \)’s, then \( \sum_{i=1}^N X_i \) is \( PQD \).

Moreover, using Property 4.4 (i), we can make explicit some links between the concepts of \( PQD \) and stop-loss ordering. This result extends Theorem 4 in Dhaene and Goovaerts (1996) (where \( \phi(x_1, x_2) = x_1 + x_2 \)).

**Proposition 5.3.** Let \( X \) be a \( PQD \) random vector valued in \( \mathbb{R}^2_+ \). Then, \( \phi(X) \preceq_2 \text{L-cx} \phi(X) \) for any function \( \phi : \mathbb{R}^2_+ \to \mathbb{R}^+ \in \mathcal{U}_2 \).
Illustration: an automobile portfolio. Let us consider a third party liability automobile portfolio generating \( N \) claims during the reference period, \( N \) being a non-negative integer-valued random variable. Usually a distinction is made among the claims according as they are concerned with physical injury or material only. We thus split the total number \( N \) of claims into \( N_1 + N_2 \), where \( N_1 \) counts the claims that have caused physical injury and \( N_2 \) those that only caused material damage. Now, let us subdivide the claim amounts into two parts according to the nature of the damage, physical and material. This yields the bivariate random vector

\[
T = \left( \sum_{i=1}^{N_1} X_i, \sum_{i=1}^{N_1} Y_i + \sum_{j=1}^{N_2} Z_j \right),
\]

where the pair \((X_i, Y_i)\) represents the amount paid by the insurer for the \( i \)th mixed claim \((1 \leq i \leq N_1)\), and \( Z_j \) is the amount paid for the \( j \)th material claim \((1 \leq j \leq N_2)\) – the claim frequencies and severities are assumed to be independent here and in the remainder. One may reasonably assume that the pairs \((X_i, Y_i)\) are PQD and independent, the \( Z_j \)'s are independent and both sequences are independent.

By Property 5.2, we get that the random vector \( T \) is also PQD, so that by (5.7), \( T^{\perp} \preceq_{\mathcal{L}_{1-cx}} T \) and by Proposition 5.3,

\[
\sum_{i=1}^{N_1}(X_i^+ + Y_i^+) + \sum_{j=1}^{N_2} Z_j \preceq_{\mathcal{L}_{2-cx}} \sum_{i=1}^{N_1}(X_i + Y_i) + \sum_{j=1}^{N_2} Z_j
\]

for the total claims. Expression (5.8) shows that the aggregate claim under the simplifying assumption of independence is smaller, in the stop-loss sense, than the actual total claim. Since most premium calculation principles agree with the stop-loss ordering (see, e.g. Goovaerts et al. (1990) and Kaas et al. (1994)), we deduce that the independence assumption for such a portfolio leads to an underestimation of the premium charged to the insured people.

When the distribution of the variables involved are known, the Fréchet bounds provide the portfolio claim amounts \( T_{\min}^{(1)} \) and \( T_{\max}^{(1)} \) such that

\[
T_{\min}^{(1)} \preceq_{\mathcal{L}_{1-cx}} T \preceq_{\mathcal{L}_{1-cx}} T_{\max}^{(1)}.
\]

The sums of the components of the vectors are then ordered in the stop-loss sense, which yields bounds on the stop-loss premium associated with the portfolio.

6. \( \preceq_{\mathcal{L}_{2-cx}} \) bounds

A problem of practical interest is how to construct bounds on quantities of the form \( E\phi(X) \) when the function \( \phi \) belongs to \( \mathcal{U}_{\mathcal{L}_{2-cx}}^S \) for some \( S \). This is the case especially in actuarial sciences where \( X \) represents a vector of two risks or two lifetimes for instance. The components of \( X \) are often interdependent but the precise form of this dependence is generally unknown which makes impossible the exact computation of \( E\phi(X) \). Of course, the actuary could make the simplifying assumption that the components of \( X \) are independent, so that he would then be able to determine \( E\phi(X) \) (at least if the marginal distributions of \( X \) are known). A crucial question, however, is to evaluate the error made by such a simplification.

When the marginal distributions of \( X \) are specified (like in life insurance for instance, \( X_1 \) and \( X_2 \) representing the lifelengths of a couple of married persons), the Fréchet bounds usually provide informative margins for any \( \phi \in \mathcal{U}_{\mathcal{L}_{2-cx}}^S \). If, moreover, \( X \) is PQD (resp. NQD), a more accurate lower (resp. upper) bound on \( E\phi(X) \) is provided by \( E\phi(X^{\perp}) \).
Suppose now that only partial information on the marginal distributions is available, in the sense that \( X_i, i = 1, 2 \), is known to belong to the class \( \mathcal{B}_i(S_i; \mu_1^{(i)}, \mu_2^{(i)}, \ldots, \mu_{s_i-1}^{(i)}) \) of all the random variables with support \( S_i \) and common \( s_i - 1 \) first moments \( \mu_1^{(i)}, \mu_2^{(i)}, \ldots, \mu_{s_i-1}^{(i)} \). If \( X \) is positively (resp. negatively) \( s \)-dependent, \( E \phi(X^{\perp}) \) is again a lower (resp. upper) bound, but the difficulty then comes from the computation of \( E \phi(X^{\perp}) \) itself. As shown by Denuit et al. (1998), given any random variable \( X \) in \( \mathcal{B}_i(S; \mu_1, \mu_2, \ldots, \mu_{s-1}) \), it is possible to determine explicitly two random variables \( X_{\min}^{(s)} \) and \( X_{\max}^{(s)} \) that are \( E \)-icx lower and upper bounds for \( X \). Therefore, by Property 4.4 (ii), we have that

\[
(X_{\min}^{(s)}, X_{\max}^{(s)}) \preceq E \text{-icx } (X_{\min}^{(s)}, X_{\max}^{(s)})
\]

so that a lower (resp. upper) bound for \( E \phi(X) \) is \( E \phi([X_{\min}^{(s)}, X_{\max}^{(s)}]) \) (resp. \( E \phi([X_{\min}^{(s)}, X_{\max}^{(s)}]) \)).

References


