Compound Poisson approximations for individual models with dependent risks

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Received 1 June 2002; received in revised form 1 September 2002; accepted 6 November 2002

Abstract

This paper shows how compound Poisson distributions can be used to approximate the distribution of the total claim amount in the context of single- or multi-class individual risk models where dependence between the contracts arises through mixtures. Some of these models are generated by Archimedean copulas, and others are seen to fall under the purview of a general multi-class shock model whose structure is both intuitive and easily tractable. A numerical study is used to illustrate the quality of the approximation as a function of the heterogeneity and the dependence in the portfolio. A theoretical result is also provided which helps to explain the effect of dependence on the total claim amount when the contracts are linked through an Archimedean copula model.

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MSC: IM11; IM12; IM30

Keywords: Compound Poisson approximation; Copula; Dependent risks; Individual risk model; Collective risk model

1. Introduction

In the classical individual risk model consisting of \( n \geq 1 \) insurance policies over a given period, the risk \( X_\ell \) corresponding to the \( \ell \)th contract is usually expressed in the form

\[
X_\ell = \begin{cases} 
B_\ell & \text{if } I_\ell = 1, \\
0 & \text{if } I_\ell = 0,
\end{cases}
\] (I)

where \( I_\ell \) is a Bernoulli random variable taking the value 1 when at least one claim is filed while the contract is valid, and \( B_\ell \) is a strictly positive random variable representing the sum of all amounts claimed in that period. The \( B_\ell \)’s are typically assumed to be independent of each other and of all the \( I_\ell \)’s.

Of particular interest is the cumulative distribution function \( F_S \) of the aggregate claim amount

\[
S = \sum_{\ell=1}^{n} X_\ell
\]
and functions thereof such as the stop-loss premium \( \pi_S(d) = E[\max(S - d, 0)] \) for \( d \geq 0 \), the value-at-risk \( \text{VaR}_\alpha = F^{-1}_S(\alpha) \) for some \( \alpha \in (0, 1) \), or the expected shortfall \( E(S | S > \text{VaR}_\alpha) \), also known as the conditional VaR.

As it is generally difficult to compute these quantities for large portfolios, \( F_S \) often needs to be approximated. When the \( X_k \)'s are stochastically independent, a useful strategy is to approach \( S \) by

\[
T = \begin{cases} 
\sum_{\ell=1}^N Z_\ell & \text{if } N > 0, \\
0 & \text{if } N = 0, 
\end{cases}
\]

where \( N \) follows a Poisson distribution with mean \( \lambda \) and \( Z_1, Z_2, \ldots \) is a sequence of independent and identically distributed random variables with common distribution function \( F_Z \). The strategy for selecting the appropriate parameters in \( T \)'s compound Poisson distribution \( F_T(\lambda, F_Z) \) varies according to the risk measure of interest. Thus if the stop-loss premium or the expected shortfall is of primary concern, then \( \lambda \) and \( F_Z \) should be chosen in such a way that \( T \) dominates \( S \) in the convex order, while if the focus is on value-at-risk, stochastic dominance of \( T \) is needed to get conservative compound Poisson approximations.

To measure the quality of the approximation of \( S \) by \( T \), three different distances between their distribution functions are typically used: the Kolmogorov metric, viz.

\[
d_K(S, T) = \sup_{x \in \mathbb{R}} |F_S(x) - F_T(x)|,
\]

the total-variation distance, viz.

\[
d_{TV}(S, T) = \sup_{A \in \mathcal{B}(\mathbb{R})} |P(S \in A) - P(T \in A)|
\]

with \( \mathcal{B}(\mathbb{R}) \) denoting the Borel sets on \( \mathbb{R} \), and the stop-loss distance, viz.

\[
d_{SL}(S, T) = \sup_{x \geq 0} |\pi_S(x) - \pi_T(x)|.
\]

In practice, however, the assumption that risks are independent is not always legitimate. For this reason, various individual risk models incorporating dependence have recently been proposed in the actuarial literature; see, among others, Bäuerle and Müller (1998), Albers (1999), Cossette et al. (in press) or Cossette et al. (2002). This, in turn, poses the problem of devising appropriate compound Poisson approximations for such models.

When the dependence between the components of a risk portfolio is local or weak, Goovaerts and Dhaene (1996) show that the total claim amount \( S \) can be approximated to a reasonable degree by a single compound Poisson variable \( T \), which is constructed as in the independent case. For mixture models, however, in which risks are conditionally independent given a (possibly multidimensional) mixing parameter, it is seen here that mixtures of compound Poisson approximations provide a simple and efficient alternative.

Two classes of mixture models for individual risks are described in Section 2, and corresponding approximations are proposed in Section 3. Since the simplicity of the models considered allows for a clear understanding of the effect of dependence and heterogeneity among risks, the impact of both factors on the quality of the approximation is explored numerically in Section 4.

### 2. Individual models with dependence among risks

A number of strategies have been proposed recently to account for dependence among risks in the individual model. They can be broadly classified in two groups: those in which the dependence is described in terms of random variables, and those where the relation between risks is expressed in terms of distribution functions. Examples of mixture models of each kind are described in turn.
2.1. A general multi-class shock model

One fruitful way to introduce dependence in a single- or multi-class risk model is in terms of shocks or catastrophes that could affect the entire portfolio or subclasses thereof. A general model of this sort is described below which includes recent proposals by Albers (1999) and Cossette et al. (2002) as special cases.

Consider a portfolio that is divided into \( m \geq 1 \) classes comprising \( n_1, \ldots, n_m \geq 1 \) contracts, and let \( X_{k\ell} \) represent the risk associated with the \( \ell \)th contract in the \( k \)th class, so that the aggregate claim amount is given by

\[
S = \sum_{k=1}^{m} n_k \sum_{\ell=1}^{n_k} X_{k\ell}.
\]

Assume that the entire portfolio can be affected by a global shock, represented by an indicator random variable \( J \), and that all risks in class \( k \) are subject to a further shock whose probability of occurrence may differ according to whether the overall catastrophe has taken place (\( J = 1 \)) or not (\( J = 0 \)). Let \( J(\alpha) \) indicate the presence or absence of a shock specific to class \( k \), given \( J = \alpha \). For fixed values of \( \alpha \) and \( \beta = J(\alpha) \in \{0, 1\} \), the risk associated with the \( \ell \)th contract in the \( k \)th class may then be modeled as in (1) by

\[
X(\alpha\beta)_{k\ell} = J(\alpha\beta)_{k\ell} B(\alpha\beta)_{k\ell},
\]

where \( J(\alpha\beta)_{k\ell} \) is a Bernoulli random variable indicating whether at least one claim was filed or not, and \( B(\alpha\beta)_{k\ell} \) is a strictly positive random variable representing the total amount claimed.

Denote \( \bar{I} = 1 - I \) for any probability or indicator function \( I \). The risk \( X_{k\ell} \) may then be represented in the form

\[
X_{k\ell} = J(J(1)_{k} X(11)_{k\ell} + \bar{J}(1)_{k} X(10)_{k\ell}) + \bar{J}(J(0)_{k} X(01)_{k\ell} + \bar{J}(0)_{k} X(00)_{k\ell}).
\]

(2)

It seems reasonable to suppose that

(a) \( J \), the \( J(\alpha) \)'s and the \( J(\alpha\beta) \)'s are mutually independent; and that

(b) the \( B(\alpha\beta) \)'s are independent of each other and of all indicator random variables.

These conditions, which will be assumed to hold throughout the paper, are satisfied in the following special cases.

Example 2.1. Eq. 2.1 of Albers (1999) defines a multi-class dependence model which may be written directly as

\[
X_{k\ell} = J(0)_{k} X(00)_{k\ell} + J(1)_{k} X(10)_{k\ell} = J(0)_{k} B(0)_{k\ell} + J(1)_{k} B(1)_{k\ell},
\]

with the above notation. This is obviously a specialization of (2) in which \( J = 0 \), so that \( J(1), J(0), J(11), J(10), B(1), B(0) \) need not be defined. While there is no provision in this model for a shock that would be common to all classes, Albers notes that when all risks are combined into a single class, \( J(10) \) can be assimilated to a global shock. This is what happens in the second example.

Example 2.2. To introduce dependence between the \( n = n_1 \) risks of the single-class model (1), Cossette et al. (2002) suppose that \( I_k \) may be written as

\[
I_k = \min(I_k + J_0, 1),
\]

in terms of mutually independent Bernoulli random variables \( J_0, \ldots, J_n \) with means \( r_0, \ldots, r_n \), respectively, chosen in such a way that \( r_0 \leq \min(q_1, \ldots, q_n) \) with

\[
q_\ell = P(I_k = 0) = P(J_0 = J_k = 0) = r_\ell, \quad 1 \leq \ell \leq n.
\]
This construction, which reduces to independence when \( r_0 = 0 \), is seen to fall under the purview of the above general model if one sets \( J = 0, J_2^{(0)} = J_0 \) and

\[
J_1^{(01)} = 1, \quad J_1^{(00)} = J_1, \quad J_2^{(01)} = r^{(00)}_1 = B_1
\]

in formula (2). For, one then finds

\[
X_{k\ell} = J_0 B_1 + J_1 J_1 B_1,
\]

which is equivalent to the representation \( X_{k\ell} = I_k B_1 \) with \( I_k \) defined as in (3). Note that \( J_1^{(11)} = J_1^{(10)} = J_1^{(01)} = J_1^{(00)} = B_1 \) and \( B_1^{(11)} \) need not be defined in this case, since \( J = 0 \). Example 2.3. When several classes are involved, Cossette et al. (2002) model the risk associated with the \( k \)th contract in the \( \ell \)th class as \( X_{k\ell} = I_k B_\ell \) with

\[
I_k = \text{min}(J_k + J_1 + J_0, 1), \tag{4}
\]

where \( J_0 \), the \( J_k \)'s and the \( J_1 \)'s are mutually independent Bernoulli random variables with mean \( r_0, r_k \) and \( r_1 \), respectively. Here, \( r_0, \ldots, r_m \) are parameters which, in broad terms, increase the dependence among the risks as they get larger. The \( I_k \)'s are independent if and only if \( r_0 = \ldots = r_m = 0 \).

Note that in view of relation (4) and the independence hypothesis on \( J_0 \), the \( J_k \) and the \( J_1 \)'s, one has

\[
q_{k\ell} = P(J_{k\ell} = 0) = P(J_k = J_1 = J_0 = 0) = r_k r_1 r_0.
\]

This multi-class model is subsumed by the general shock model (2) when

\[
J = J_0, \quad J_1^{(1)} = 1, \quad J_1^{(0)} = J_1, \quad J_2^{(1)} = 1, \quad J_2^{(0)} = J_2 = J_k,
\]

and

\[
B_1^{(11)} = B_1^{(10)} = B_1^{(01)} = B_1 = B_\ell.
\]

Of course, there are again many variables that need not be defined, such as \( J_\ell^{(11)} \), for instance.

In the multi-shock model (2), the distribution function \( F_{X_{k\ell}} \) of \( X_{k\ell} \) is easily derived from those of the \( X_{\ell k}^{(0\ell)} \)'s and of the \( B_{1\ell}^{(0\ell)} \)'s. Indeed, if

\[
E(J) = 1, \quad E(J_1^{(0)}) = q_{1\ell} \quad \text{and} \quad E(J_2^{(0)}(\bar{J}_1)) = r_{2\ell
\]

for every choice of \( 1 \leq \ell \leq n_2 \) and \( 1 \leq k \leq m \), a straightforward conditioning argument exploiting conditions (a) and (b) above implies that

\[
F_{X_{k\ell}}(x) = q_{k\ell} + r_{k\ell}^{(1)} + F_{X_{k\ell}}(x) + r_{k\ell}^{(1)} + F_{X_{k\ell}}(x) + r_{k\ell}^{(1)} + F_{X_{k\ell}}(x)
\]

and a simple rearrangement of the terms then yields

\[
F_{X_{k\ell}}(x) = q_{k\ell} + \bar{r}_{k\ell}^{(1)} + F_{X_{k\ell}}(x) + r_{k\ell}^{(1)} + F_{X_{k\ell}}(x) + \bar{r}_{k\ell}^{(1)} + F_{X_{k\ell}}(x) + \bar{r}_{k\ell}^{(1)} + F_{X_{k\ell}}(x), \quad x \geq 0,
\]

where

\[
r_{k\ell} = r_{k\ell}^{(1)} + r_{k\ell}^{(1)} + r_{k\ell}^{(1)} + r_{k\ell}^{(1)} + r_{k\ell}^{(1)} + r_{k\ell}^{(1)} + r_{k\ell}^{(1)} + r_{k\ell}^{(1)} + r_{k\ell}^{(1)} = P(X_{k\ell} \neq 0). \tag{5}
\]
Accordingly, each risk \( X_{k\ell} \) may be expressed in the form (1) as

\[
X_{k\ell} = \begin{cases} 
B_{k\ell} & \text{if } I_{k\ell} = 1, \\
0 & \text{if } I_{k\ell} = 0,
\end{cases}
\]

in terms of an indicator random variable \( I_{k\ell} \) with \( P(I_{k\ell} = 1) = q_{k\ell} \) defined as in Eq. (5), and a strictly positive random variable \( B_{k\ell} \) with mixture distribution

\[
F_{B_{k\ell}} = \frac{r_{k}^{(1)}(1)}{q_{k\ell}} F_{a_{k1}}^{(1)} + \frac{r_{k}^{(1)}(0)}{q_{k\ell}} F_{a_{k2}}^{(0)} + \frac{r_{k}^{(0)}(1)}{q_{k\ell}} F_{a_{k3}}^{(1)} + \frac{r_{k}^{(0)}(0)}{q_{k\ell}} F_{a_{k4}}^{(0)}.
\]

In this model, however, the \( X_{k\ell} \)'s are generally far from being independent, which makes it hard to evaluate directly the distribution \( F_S \) of the total claim amount \( S \) of the portfolio. As will be seen in Section 3, a handy way of circumventing this problem lies in the fact that under conditions (a) and (b), relation (2) induces a mixture structure for \( S \). In other words, there exists a latent random vector \( \Theta \) with distribution \( M \) for which \( F_S \) may be expressed in the form

\[
F_S(x) = \int F_{S(\theta)}(x) \, dM(\theta), \tag{6}
\]

where \( S(\theta) \) is distributed as \( S \) given \( \theta = \bar{\theta} \).

To be specific, let \( \theta = (\theta_0, \ldots, \theta_m) \) take values in \( \{0, 1\}^{m+1} \) in such a way that

\[
P(\theta_0 = 0) = P(J = 0) = r^{a_{11}}
\]

and

\[
P(\theta_0 = 1, \ldots, \theta_m = 1) = P(J^{(a)} = 1) = \prod_{k=1}^{m} (r_{a_{k1}}^{(1)} g_{a_{k1}}^{(1)})^{1-a_{k1}}
\]

for all \( a_1, a_2, \ldots, a_m \in [0, 1] \). For \( \theta = a \), write \( X_{k\ell}^{(a)} = X_{k\ell}^{(a_{k\ell})} \) and let

\[
S_{k\ell}^{(a)} = \sum_{\ell=1}^{n_{k\ell}} X_{k\ell}^{(a_{k\ell})} \quad \text{and} \quad S^{(a)} = \sum_{k=1}^{n} S_{k\ell}^{(a)}.
\tag{7}
\]

Then in view of (2), one has

\[
F_S(x) = P \left( \sum_{k=1}^{n} \sum_{\ell=1}^{n_{k\ell}} X_{k\ell} \leq x \right)
= \sum_{a \in \{0, 1\}^{m+1}} \sum_{\theta \in \{0, 1\}^{m+1}} P(\theta = (a, \beta_1, \ldots, \beta_m))
= \sum_{a \in \{0, 1\}^{m+1}} P(\theta = a) \int F_{S(\theta)}(x) \, dM(\theta),
\]

which is the same as (6) for \( \theta \) discrete.

The advantage of this representation is that for given \( \theta \), the \( S_{k\ell}^{(a)} \)'s are mutually independent and each of them is a finite sum of mutually independent random variables.
2.2. Single-class risk models based on copulas

Eq. (2), and relations (3) and (4) which stem from it in special cases, introduce dependence between risks through shocks represented by indicator variables. In the single-class portfolio problem, where each risk may be written as

\[ X_\ell = I_\ell B_\ell \]

with Bernoulli random variables \( I_\ell \) with expectation \( q_\ell \), another way of modeling dependence among the components of the vector \( I = (I_1, \ldots, I_n) \) is to write their joint cumulative distribution function as

\[ F_I(i_1, \ldots, i_n) = C\{ F_{I_1}(i_1), \ldots, F_{I_n}(i_n) \} \]

where \( F_{I_\ell}(t) = P(I_\ell \leq t) \) and \( C : [0, 1]^n \to [0, 1] \) is a copula, that is, the restriction to \([0, 1]^n\) of a cumulative distribution function with uniform marginals on the unit interval. The advantage of this approach, considered by Cossette et al. (2002), is that the distributions \( F_{I_\ell} \) of the occurrence variables \( I_\ell \) and their joint dependence structure can then be modeled separately.

Writing \( F_I \) in the form (8) is not restrictive per se, since it was shown by Sklar (1959) that every multivariate distribution function may be written this way for some copula \( C \). What is an assumption, however, is to suppose that \( C \) is of a specific form or that it belongs to a given class \( C \) of copulas, such as

\[ C_\gamma (u_1, \ldots, u_n) = \left( 1 - \gamma \sum_{\ell=1}^n u_\ell + \frac{\gamma \min(u_1, \ldots, u_n)}{\gamma} \right)^{-1/\gamma}, \quad \gamma > 0 \]

known as the model of Clayton (1978).

The reader may refer to the review article by Frees and Valdez (1998) or to the books by Hutchinson and Lai (1990), Joe (1997), Nelsen (1999) and Drouet-Marie and Kotz (2001) for a large selection of models of this sort.

Of particular interest here are copula-based dependence models that induce a mixing structure for the total claim distribution \( F_S \). Two classes of this type are described below: the elementary Fréchet mixture model, and the much richer structure associated with the so-called Archimedean copulas (Genest and MacKay, 1986). By definition, the latter are expressible in the form

\[ C(u_1, \ldots, u_n) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_n)) \]

for some function \( \phi : (0, 1) \to [0, \infty) \) such that \( \phi(1) = 0 \) and

\[ (-1)^\ell \frac{d^\ell}{dt^\ell} \phi^{-1}(t) \geq 0, \quad 1 \leq \ell \leq n. \]

Popular choices for actuarial applications include

(a) \( \phi_\gamma(t) = (t^{-\gamma} - 1)/\gamma \) with \( \gamma > 0 \), which yields the Clayton model defined above;

(b) \( \phi_\gamma(t) = [\log(1/t)]^\gamma \) with \( \gamma \geq 1 \), which generates Gumbel’s family of (extreme value) copulas;

(c) \( \phi_\gamma(t) = \log(e^{t^\gamma} - 1)/(e^{t^\gamma} - 1) \) with \( 0 < \gamma < \infty \), which corresponds to the positively associated members of Frank’s family.

See Genest et al. (1998) for a three-parameter family of Archimedean copulas that encompasses all these models.

Example 2.4. Suppose that the underlying copula of the vector \( I \) is of the form

\[ C_\gamma(u_1, \ldots, u_n) = (1 - \gamma)u_1 \cdots u_n + \gamma \min(u_1, \ldots, u_n) \]

for some \( 0 \leq \gamma \leq 1 \), so that \( I_1, \ldots, I_n \) are mutually independent with probability \( 1 - \gamma \), but comonotonic in the sense of Wang and Dhaene (1997) with probability \( \gamma \). Further assume without loss of generality that the \( I_\ell \)'s are labeled in such a way that \( q_\ell = P(I_\ell = 1) \leq P(I_{\ell'} = 1) = q_{\ell'} \) when \( \ell \leq \ell' \). Then

\[ I_\ell = F_{I_\ell}^{-1}(V), \quad 1 \leq \ell \leq n \]
for some random variable $V$ with uniform distribution on $[0, 1]$, so that

$$I_1 = \cdots = I_{\ell-1} = 0 \quad \text{and} \quad I_{\ell} = \cdots = I_n = 1$$

if and only if (up to a set of measure zero) $1 - q_\ell \leq V \leq 1 - q_{\ell-1}$ for $1 \leq \ell \leq n$ (with $q_0 = 0$). Furthermore, $I_1 = \cdots = I_n = 0$ when $0 \leq V \leq 1 - q_n$. Conditionally on the $I_\ell$’s being comonotonic, therefore, the distribution of $S$ is then given by

$$F_{\text{com}}(s) = \sum_{\ell=1}^{n} (q_\ell - q_{\ell-1}) F_{B_{\ell}}(s) + (1 - q_n) 1_{[0, \infty)}(s)$$

with $B_{\ell} = B_1 + \cdots + B_{\ell}$, $1 \leq \ell \leq n$. Unconditioning, one can then write

$$F_S(s) = (1 - \gamma) F'_S(s) + \gamma F_{\text{com}}(s)$$

where $S' = X'_1 + \cdots + X'_n$ is the sum of mutually independent random variables $X'_\ell$ whose univariate distributions are the same as those of the $X_\ell$’s.

The above model can be viewed as a special case of the previous shock model with a single class in which the indicator random variables $J(\alpha)_{\ell}$, $1 \leq \ell \leq n$, are either comonotonic or mutually independent according as a global shock $J$ with $P(J = 1) = \gamma$ occurs ($\alpha = 1$) or not ($\alpha = 0$). The following example, however, is of a different type.

**Example 2.5.** Suppose that the joint distribution of the vector $I$ is of the form (8) with Archimedean copula $C$ generated by $\phi$. Further assume that condition (10) is verified for all integers $n \geq 1$, so that $\phi^{-1}$ is completely monotone and hence the Laplace transform of a distribution $M$ whose support is included in $[0, \infty)$. Following Marshall and Olkin (1988), $F_I(i_1, \ldots, i_n)$ may then be viewed as a mixture of powers, that is, it can be written in the form

$$F_I(i_1, \ldots, i_n) = \int_0^{\infty} \prod_{\ell=1}^{n} F_{I_\ell}(i_\ell) dM(\theta), \quad (11)$$

where $F_{I_\ell}(t) = \exp\{-\phi[I_\ell(t)]\}$ is the cumulative distribution function of a Bernoulli random variable with expectation $1 - \exp[-\phi(1 - q_\ell)]$, $1 \leq \ell \leq n$. In other words, $\theta$ is a latent variable conditional upon which the $I_\ell$’s are mutually independent Bernoulli random variables with mean

$$E(I_\ell|\theta = \theta) = 1 - \exp[-\theta \phi(1 - q_\ell)] \equiv q_\ell\theta, \quad 1 \leq \ell \leq n. \quad (12)$$

For given $\theta > 0$, let $I_{\ell\theta}$, $1 \leq \ell \leq n$, be mutually independent indicator random variables such that $E(I_{\ell\theta}) = q_\ell\theta$. Assuming, as in the standard single-class model, that the $B_{\ell}$’s are independent of each other and of all the indicators, the components

$$X_\ell^{(\theta)} = \begin{cases} B_\ell & \text{if } I_{\ell\theta} = 1, \\ 0 & \text{if } I_{\ell\theta} = 0, \end{cases}$$

of the sum $S^{(\theta)} = X_1^{(\theta)} + \cdots + X_n^{(\theta)}$ are then mutually independent, so that $F_S$ admits the mixture representation (6).

For an example of application of model (11), see Cossette et al. (in press).
2.3. Multi-class risk models based on copulas

To define a multi-class extension of the Archimedean model of Section 2.2, suppose that the joint distribution of the \((n_1 + \cdots + n_m)\)-dimensional vector \(I = (i_1, \ldots, i_m)\) of occurrence random variables is given by

\[
F_I(i_1, \ldots, i_m) = \int_0^\infty \cdots \int_0^\infty \prod_{k=1}^m \left( F_{X_k}(i_k) \right)^{nk} \, dM(\theta_1, \ldots, \theta_m),
\]

where \(M\) is the \(m\)-variate distribution function of the latent vector \(\Theta = (\Theta_1, \ldots, \Theta_m) \in [0, \infty]^m\) and \(F_{X_k}(t) = \exp\left(-\phi_k(t)\right)\) is the cumulative distribution function of a Bernoulli random variable with expectation \(1 - \exp{-\phi_k(1 - q_k)}\) with \(1 \leq \ell \leq n_k\) and \(1 \leq k \leq m\). Conditional on the value of \(\Theta\), the \(i_k\)'s are then mutually independent Bernoulli random variables with

\[
E(I_k | \Theta = \theta) = E(I_k | \Theta_k = \theta_k) = 1 - \exp{-\theta_k \phi_k(1 - q_k)} = q_k \theta_k.
\]

For all \(1 \leq k \leq m\) and \(1 \leq \ell \leq n_k\), let \(I^{(y)}_{k\ell}\) be mutually independent indicator random variables with mean \(q_k \theta_k\), and assume as before that the \(B_{k\ell}\)'s are independent among themselves and from all indicators. Conditional on \(\Theta = \theta\), the sums \(S^{(y)}_k\) and \(S^{(y)}\) defined as in (7), but with

\[
X^{(y)}_{k\ell} = X^{(y)}_k = I^{(y)}_{k\ell} \cdot B_{k\ell}
\]

are then composed of mutually independent terms. Consequently, the total claim distribution \(F_S\) may again be represented in the form (6).

3. Approximation strategy and applications

In the individual and collective risk models described in Section 2, dependence is introduced via a (possibly multivariate) mixing variable \(\Theta\) which is either latent, as in Archimedean copulas, or explicitly represented by the indicators of the common and class shocks in the general construction of Section 2.1.

In all cases, it was seen that the total claim distribution could be expressed in the form

\[
F_S(x) = \int F_S(y) \, dM(\theta)
\]

involving the distribution function of a double sum

\[
S^{(y)} = \sum_{k=1}^m S^{(y)}_k = \sum_{k=1}^m \sum_{\ell=1}^{n_k} X^{(y)}_{k\ell}
\]

of random variables \(X^{(y)}_{k\ell}\) that are mutually independent given \(\Theta = \theta\). Furthermore, the latter may all be expressed in the form

\[
X^{(y)}_{k\ell} = I^{(y)}_{k\ell} \cdot B_{k\ell}
\]

as in (1). To evaluate \(F_S\), therefore, one could seek (if necessary) a suitable discretization of the \(B_{k\ell}\)'s and use, say, the algorithms of DePril that can be found in the book by Rolski et al. (1999, Chapter 4).

As a natural alternative often considered in actuarial science, compound Poisson approximations could be developed for \(F_S\) by exploiting standard results to approach each sum \(S^{(y)}\) of independent risks given \(\Theta = \theta\) by a Poisson variable \(T^{(y)}\). See the authoritative text by Barbour et al. (1992) for a survey of the topic, and Rolski et al. (1999, Chapter 4), among other, for actuarial applications.
Upon unconditioning, the fact that
\[ d(S, T) \leq \int_0^\infty d(S^{(0)}, T^{(0)}) dM(\theta) \] (14)
can then be used to derive bounds on the overall quality of the approximation for specific choices of the distance \(d\). Witte (1990) gives a series of such bounds for \(d = d_K, d_{TV}\) or \(d_{SL}\), and the conditions under which they apply. Other key references concerning compound Poisson approximations include Serfozo (1986, 1988) and Vellaisamy and Chandhuri (1999).

Detailed descriptions of this Poisson approximation strategy are given below, both for the general multi-class model and the mixture models based on Archimedean copulas.

3.1. Poisson approximation for the multi-class shock model

Suppose that the risk \(X_{k\ell}\) associated with the \(\ell\)th contract in the \(k\)th class of a portfolio can be affected as in (2) by a random global catastrophe \(J\) and a class-specific shock \(J(\alpha)\) whose probability of occurrence depends on the value of the indicator \(J = \alpha \in \{0, 1\}\). It was seen in Section 2.1 that given a vector \(\Theta = \theta = (\alpha, \beta_1, ..., \beta_m)\) whose components indicate whether the global catastrophe and the \(m\) class-specific shocks occurred or not, the random sums \(S(\Theta)\) and \(S^{(0)}_1, ..., S^{(0)}_m\) defined in (7) consist of mutually independent terms.

In particular, for each \(1 \leq k \leq m\), \(S^{(0)}_k = S^{(0)}_{\alpha \beta_k}\) may thus be approximated by a Poisson variable \(T^{(0)}_{\alpha \beta_k}\) chosen in such a way that either
\[ E(T^{(0)}_{\alpha \beta_k}) = E(S^{(0)}_{\alpha \beta_k}) \] (15)
or
\[ P(T^{(0)}_{\alpha \beta_k} = 0) = P(S^{(0)}_{\alpha \beta_k} = 0) \] (16)
so that \(T^{(0)}_{\alpha \beta_k}\) dominates \(S^{(0)}_{\alpha \beta_k}\) in the convex or in the stochastic dominance order, respectively.

A Poisson approximation of \(S\) is then given by \(T = JT^{(1)} + JT^{(0)}\) with
\[ T^{(\alpha)} = \sum_{k=1}^m j^{(\alpha)}_k T^{(1)}_{\alpha \beta_k} + \sum_{k=1}^m j^{(\alpha)}_k T^{(0)}_{\alpha \beta_k}, \quad \alpha = 0, 1. \] (17)

An evaluation of the right-hand side of (14) now yields
\[ \sum_{\alpha=0}^1 \sum_{\beta=0}^1 \sum_{\beta' \neq \beta} \sum_{\beta'' \neq \beta'} \frac{1}{r \gamma^{\alpha \beta \beta'} \gamma^{\alpha \beta' \beta''}} d \left( \sum_{k=1}^m S^{(\alpha \beta_k)} - \sum_{k=1}^m T^{(\alpha \beta_k)} \right) \]
which provides an upper bound on the quality of the above approximation, whether \(d\) stands for the Kolmogorov, the total-variation or the stop-loss distance. A looser, but possibly easier bound to compute can be derived from the inequality
\[ d \left( \sum_{k=1}^m S^{(\alpha \beta_k)} - \sum_{k=1}^m T^{(\alpha \beta_k)} \right) \leq \sum_{k=1}^m d \left( S^{(\alpha \beta_k)}, T^{(\alpha \beta_k)} \right), \]
which obtains because for fixed \(\alpha\), the \(S^{(\alpha \beta_k)}\)'s and the \(T^{(\alpha \beta_k)}\)'s form a collection of \(2m\) mutually independent random variables.
The following proposition summarizes the above developments.

**Proposition 3.1.** In the general multi-class model (2), a Poisson approximation to the total claim amount $S$ is provided by $T = JT^{(1)} + \bar{J}T^{(0)}$ with $T^{(0)}$ defined as in (17) in terms of Poisson random variables $T^{(0)}_{\alpha\beta k}$ chosen as per relations (15) or (16). An upper bound on the quality of this approximation is given by

$$\sum_{\alpha=0}^{m} \sum_{\beta=0}^{m} \cdots \sum_{\ell=0}^{m} r^{(\alpha)}_{1} \prod_{k=1}^{r^{(\alpha)}_{1}} \left( r^{(\alpha)}_{1} \beta k \right) \bar{r}^{(\alpha)}_{1} \left( r^{(\alpha)}_{1} - \beta k \right) \sum_{k=1}^{m} d(S^{(\alpha\beta k)}_1, T^{(\alpha\beta k)}_1),$$

where $d$ represents either the Kolmogorov, total-variation or stop-loss distance.

**Example 3.1.** In the single-class model of Albers (1999) described in Example 2.1, one has $J = 0$ and hence $r = 0$. An upper bound on the quality of the approximation $T = T^{(0)} = T^{(0)}_1 T^{(0)}_1 + \bar{J}T^{(0)}_1$ of $S$ is thus given by

$$d_{TV}(S(T^{(0)}_1), T(T^{(0)}_1)) \leq \sum_{\ell=1}^{n_1} r^{(0)}_{1} \prod_{k=1}^{r^{(0)}_{1}} \left( r^{(0)}_{1} \beta k \right) \bar{r}^{(0)}_{1} \left( r^{(0)}_{1} - \beta k \right) E(B^{(0)}_{1\ell}),$$

and

$$d_{SL}(S(T^{(0)}_1), T(T^{(0)}_1)) \leq \sum_{\ell=1}^{n_1} r^{(0)}_{1} \prod_{k=1}^{r^{(0)}_{1}} \left( r^{(0)}_{1} \beta k \right) \bar{r}^{(0)}_{1} \left( r^{(0)}_{1} - \beta k \right) E(B^{(0)}_{1\ell}).$$

Using these inequalities in bound (18) yields

$$d_{TV}(S, T) \leq r^{(0)}_{1} \prod_{k=1}^{r^{(0)}_{1}} \left( r^{(0)}_{1} \beta k \right) \bar{r}^{(0)}_{1} \left( r^{(0)}_{1} - \beta k \right) E(B^{(0)}_{1\ell}),$$

and

$$d_{SL}(S, T) \leq r^{(0)}_{1} \prod_{k=1}^{r^{(0)}_{1}} \left( r^{(0)}_{1} \beta k \right) \bar{r}^{(0)}_{1} \left( r^{(0)}_{1} - \beta k \right) E(B^{(0)}_{1\ell}).$$

as upper bounds for the quality of the proposed Poisson approximation to the total claim amount in the model of Albers (1999).
which can then be approximated by \( T = T_0 + \hat{\beta} T_{(1)} + \hat{\beta} T_{(0)} \) with \( T_{(1)} = B_1 + \cdots + B_k \) while \( T_{(0)} \) is chosen to match either the mean of \( B_1 R_1 + \cdots + B_k R_k \) or its probability of being equal to zero. In this case, one may take \( \delta T_{(1)} = 0 \) and \( \delta T_{(0)} = \tilde{E}(\hat{\beta}) \) in the bound (18). Proceeding as in the previous example, therefore, upper bounds on the quality of this approximation, based on Gerber’s result, would take the form

\[
d_{TV}(S, T) \leq \hat{r}_1(0) \sum_{\ell=1}^{m} (J_{1\ell}^{(0)})^2 E(B_1).
\]

and

\[
d_{ST}(S, T) \leq \hat{r}_1(0) \sum_{\ell=1}^{m} (J_{1\ell}^{(0)})^2 E(B_1).
\]

**Example 3.3.** In the multi-class model of Cossette et al. (2002), one has \( J = J_0, \ f_1^{(1)} = 1, \ f_1^{(0)} = J_0, \ J_1^{(1)} = 1, \ J_1^{(0)} = 1, \ J_2^{(1)} = J_1, \) and \( B_1^{(1)} = B_1^{(0)} = B_1^{(00)} = B_1 \) for all \( 1 \leq \ell \leq n_k \) and \( 1 \leq k \leq m \). The total claim amount may thus be written as

\[
S = J_0 \sum_{k=1}^{n_k} \sum_{\ell=1}^{m} B_{1\ell} + \hat{\beta}_0 \sum_{k=1}^{n_k} \left( J_0 \sum_{\ell=1}^{m} B_{1\ell} + \tilde{E}(\hat{\beta}_k) \right).
\]

It can be approximated by

\[
T = J_0 \sum_{k=1}^{n_k} \sum_{\ell=1}^{m} B_{1\ell} + \hat{\beta}_0 \sum_{k=1}^{n_k} \left( J_0 \sum_{\ell=1}^{m} B_{1\ell} + \tilde{E}(\hat{\beta}_k) \right)
\]

in which \( T_{(0)} \) is chosen in such a way that either

\[
E(T_{(0)}) = E \left( \sum_{k=1}^{n_k} J_0 B_{1\ell} \right) \quad \text{or} \quad P(T_{(0)} = 0) = \sum_{k=1}^{n_k} J_0 B_{1\ell} = 0
\]

Since in this approximation \( J_0^{(1)} = J_0^{(0)} \) and \( T^{(1)} \) is distributed as \( S \) given \( J_0 = 1 \), a bound on the quality of this approximation is then of the form

\[
\tilde{r} \sum_{\beta_1=0}^{1} \cdots \sum_{\beta_k=0}^{1} \left( \prod_{\ell=1}^{m} (J_{1\ell}^{(0)})^{1-\beta_\ell} (J_{1\ell}^{(0)})^{\beta_\ell} \right) \sum_{k=1}^{n_k} \sum_{\ell=1}^{m} \tilde{E}(\hat{\beta}_k).
\]

In view of Gerber’s result, one can see that

\[
d_{TV}(S^{(00)}, T^{(00)}), = \sum_{\ell=1}^{m} (J_{1\ell}^{(0)})^2 E(B_{1\ell})
\]

and

\[
d_{ST}(S^{(00)}, T^{(00)}), = \sum_{\ell=1}^{m} (J_{1\ell}^{(0)})^2 E(B_{1\ell}).
\]

Consequently, the upper bound of \( d_{TV}(S, T) \) is given by

\[
\tilde{r} \sum_{\beta_1=0}^{1} \cdots \sum_{\beta_k=0}^{1} \left( \prod_{\ell=1}^{m} (J_{1\ell}^{(0)})^{1-\beta_\ell} (J_{1\ell}^{(0)})^{\beta_\ell} \right) \sum_{k=1}^{n_k} \sum_{\ell=1}^{m} \tilde{E}(\hat{\beta}_k).
\]
while the upper bound of \( d_{\text{SL}}(S, T) \) equals
\[
\tilde{r} \sum_{k=1}^{n} \sum_{k=1}^{n} \left\{ \prod_{i=1}^{n} \left( r(i) \right)_{\frac{1}{k}} - \left( r(k) \right)_{\frac{1}{i}} \right\} E(B_k).
\]

3.2. Poisson approximation for the single-class Archimedean model

Suppose that the risks \( X_{\ell} = I_{\ell} B_{\ell} \) in a single-class model are such that the \( B_{\ell} \)'s are independent of the \( I_{\ell} \)'s, and the latter are Bernoulli random variables with \( E(I_{\ell}) = q_{\ell} \) and copula
\[
C(u_1, \ldots, u_n) = \phi^{-1} \left\{ \phi(u_1) + \cdots + \phi(u_n) \right\}
\]
for some completely monotonic generator \( \phi : (0, 1) \rightarrow [0, \infty) \) whose inverse \( \phi^{-1} \) is the Laplace transform of a distribution \( M \) whose support is a subset of \([0, \infty)\).

It was shown in Example 2.5 that conditional on a realization \( \theta \) of the latent variable \( \Theta \) with distribution \( M \), the total claim amount \( S = \sum_{\ell=1}^{n} I_{\ell} B_{\ell} \) is distributed as \( S(\theta) = \sum_{\ell=1}^{n} I_{\ell} \theta B_{\ell} \), where the \( I_{\ell}\theta \)'s are mutually independent Bernoulli random variables with mean \( E(I_{\ell}\theta) = q_{\ell}\theta \) given in (12).

Now suppose that an approximation is sought for the stop-loss premium or expected shortfall associated with the portfolio \( S \). For each fixed \( \theta \), one could then approach \( S(\theta) \) by a compound Poisson random variable \( T(\theta) \) with parameters
\[
q_{\theta} = \sum_{\ell=1}^{n} q_{\ell}\theta \quad \text{and} \quad F_{\theta} = \sum_{\ell=1}^{n} q_{\ell}\theta F_{B_{\ell}},
\]
so that \( E(T(\theta)) = E(S(\theta)) \). The mixture random variable \( T \) with distribution function
\[
F_t(t) = \int_{\theta} \mathbb{P}(T(\theta) \leq t) dM(\theta)
\]
(19)
is then an approximation of \( S \) satisfying \( E(T) = E(S) \). Examples of application of inequality (14) are given below for this approximation when \( d \) is either the total-variation or stop-loss distance. Similar results could be obtained using other bounds or a different approximation \( T \) of \( S \) ensuring that \( P(T = 0) = P(S = 0) \).

Example 3.4. When \( d \) is either the total-variation or the stop-loss distance, the same result of Gerber (1984) mentioned above implies that
\[
d_{\text{TV}}(S(\theta), T(\theta)) \leq \sum_{\ell=1}^{n} q_{\ell}\theta^2 \quad \text{and} \quad d_{\text{SL}}(S(\theta), T(\theta)) \leq \sum_{\ell=1}^{n} q_{\ell}\theta^2 E(B_{\ell}),
\]
where \( q_{\theta} \) is defined as in (12). In view of (14) and the fact that \( \phi^{-1}(t) = E(e^{-t\phi}) \), it follows that
\[
d_{\text{TV}}(S, T) \leq \int_{0}^{\infty} d_{\text{TV}}(S(\theta), T(\theta)) dM(\theta) \leq \sum_{\ell=1}^{n} q_{\ell}^2 \theta^2 \sum_{\ell=1}^{n} \phi^{-1}(\theta) d\phi - 1 - 2q_{\ell} \theta + q_{\ell}^2 \theta^2.
\]
Similarly, one has
\[
d_{\text{SL}}(S, T) \leq \sum_{\ell=1}^{n} \phi^{-1}(2\phi(1 - q_{\ell})) + 2q_{\ell} - 1 \theta^2 E(B_{\ell}).
\]
Of course, both these bounds immediately reduce to those initially found by Gerber (1984) for independent risks when \( \phi(t) = \log(1/t) \) is the generator of the independence distribution.

As a concrete illustration, suppose that the underlying copula of the vector \( I \) of occurrence variables is from the Gumbel family with generator \( \phi_{\gamma}(t) = \log(1/t) \) and parameter \( \gamma \geq 1 \). One may then conclude that

\[
\text{d}_{TV}(S, T) \leq \sum_{\ell=1}^{n} \left( (1 - q_\ell)^{1/\gamma} + 2q_\ell - 1 \right),
\]

which reduces to Gerber’s original bound when \( \gamma = 1 \) (the case of independence) and increases with \( \gamma \) until it approaches \( q_1 + \cdots + q_n \) as \( \gamma \to \infty \), that is, when the \( I_\ell \)'s become comonotonic.

**Example 3.5.** When the \( B_\ell \)'s are independent and identically distributed, Goovaerts and Dhaene (1996) show that

\[
\text{d}_{TV}(S(\theta), T(\theta)) \leq 1 - e^{-q_{\theta}} \sum_{\ell=1}^{n} q_{\ell\theta},
\]

(see also Barbour and Hall, 1984). In this special case, therefore, it also follows from (14) that

\[
\text{d}_{TV}(S, T) \leq \sum_{\ell=1}^{n} \int_{0}^{\infty} q_{\ell\theta} \left( 1 - e^{-q_{\theta}} \right) \text{d}M(\theta).
\]

### 3.3. Poisson approximation for the multi-class Archimedean model

When the dependence between the multivariate indicator random vectors \( I_1, \ldots, I_m \) is modeled through Eq. (13), it was shown in Section 2.3 that conditionally on the value of a latent vector \( \Theta \), the total claim amount may be written as

\[
S(\theta) = \sum_{k=1}^{m} \sum_{\ell=1}^{n_k} I^{(\theta)}_{k\ell} B_{k\ell}
\]

in terms of mutually independent Bernoulli random variables \( I^{(\theta)}_{k\ell} \) with mean \( q_{k\ell}(\theta) \).

The approximation of each of these \( S^{(\theta)} \)'s by a compound Poisson random variable \( T^{(\theta)} \) with parameters

\[
q_{\theta} = \sum_{k=1}^{m} \sum_{\ell=1}^{n_k} q_{k\ell}(\theta) \quad \text{and} \quad F_{Z\theta} = \sum_{k=1}^{m} \sum_{\ell=1}^{n_k} \frac{q_{k\ell}(\theta)}{q_{\theta}} \text{d}F_{B_{k\ell}}
\]

leads once again to a mixture random variable \( T \) with distribution (19) such that \( E(T) = E(S) \). Proceeding exactly as in the previous section, one can see for instance that for such an approximation

\[
\text{d}_{TV}(S, T) \leq \sum_{k=1}^{m} \sum_{\ell=1}^{n_k} \left( 2q_k(1 - q_{k\ell}) + 2q_{k\ell} - 1 \right)
\]

and

\[
\text{d}_{SL}(S, T) \leq \sum_{k=1}^{m} \sum_{\ell=1}^{n_k} \left( 2q_k(1 - q_{k\ell}) + 2q_{k\ell} - 1 \right) E(B_{k\ell}).
\]
If in addition the $B_k \ell$’s are identically distributed, one has also

$$d_{TV}(S, T) \leq m \sum_{k=1}^{n} \sum_{\ell=1}^{s} \int_0^{\infty} \frac{1 - e^{-q k \ell \theta}}{q \ell} dM(\theta).$$

Similar bounds could obviously be devised for a different approximation $T$ satisfying $P(T = 0) = P(S = 0)$. In addition, formula (14) would also be applicable to the more sophisticated bounds surveyed by Witte (1990), under whatever particular conditions make the latter valid.

4. Effect of heterogeneity and dependence on the quality of the approximations

In order to quantify the effect of dependence on the quality of the Poisson approximation, a fictitious portfolio was considered which consisted of a single class comprising 1000 contracts. To introduce heterogeneity, this portfolio was divided into 100 subgroups labeled $(k_1, k_2)$ with $k_1, k_2 \in \{1, \ldots, 10\}$. It was assumed that each of the 10 risks in subgroup $(k_1, k_2)$ could be expressed as a product

$$X(k_1, k_2, \ell) = I(k_1, k_2) \ell B(k_1, k_2) \ell,$$

of two independent random variables, namely a Bernoulli random variable $I(k_1, k_2) \ell$ with

$$P(I(k_1, k_2) \ell = 1) = 0.001 + 0.0005 (q(k_1 - 1))^2,$$

and a geometric random variable $B(k_1, k_2) \ell$ distributed on the integers $\{1, 2, \ldots\}$ with mean

$$E(B(k_1, k_2) \ell) = 2 + b(k_2 - 1).$$

Here, $q$ and $b$ are arbitrary non-negative parameters ranging in $[0, 4]$ and $[0, 2]$, respectively.

To induce dependence between the contracts, a single-class risk model based on a Frank copula was used. In other words, the dependence between the $n = 1000$ indicator random variables $I(k_1, k_2) \ell$ was assumed to be of the form (8) with Archimedean copula $C$ of the type (9) with generator

$$\phi_{\gamma}(t) = \log \left\{ \frac{e^{-\gamma t} - 1}{e^{-\gamma t} - 1} \right\}, \quad \gamma > 0.$$

The statistical properties of this class of multivariate copulas were studied by Nelsen (1986) and Genest (1987), who show that the parameter $\gamma$ is related to Kendall’s coefficient of concordance between any two components (that is, between any two $I(k_1, k_2) \ell$’s) via

$$\tau(\gamma) = 1 + \frac{4}{\gamma} \left( \frac{1}{\gamma} \int_0^\infty \frac{t}{e^t - 1} dt - 1 \right).$$

Three choices of $\gamma$ were considered that correspond to independence ($\tau = 0$, $\gamma = 0$), weak dependence ($\tau = 1/8$, $\gamma \approx 1.13956$), and moderate dependence ($\tau = 1/4$, $\gamma \approx 2.37201$) between the risks. An advantage of working with Frank’s copula with parameter $\gamma > 0$ is that its associated mixture distribution $M$ is the discrete logarithmic distribution with probability function

$$P(\Theta = \theta) = \frac{(1 - e^{-\gamma})^\theta}{\theta !}, \quad \theta \in \{1, 2, \ldots\}.$$

In this special case, therefore, it is actually possible to determine

$$F_\Theta(x) = \sum_{\theta=1}^{\infty} F_{\Theta,\theta}(x) P(\Theta = \theta).$$
Table 1  
Exact and compound Poisson-based approximation for the stop-loss premium corresponding to a single-class portfolio comprising 1000 heterogeneous contracts whose occurrence random variables are independent ($\tau = 0$) or stochastically related ($\tau = 1/8$ or $1/4$) through a Frank copula model

<table>
<thead>
<tr>
<th>$x$</th>
<th>$r = 0$</th>
<th>$r = 1/8$</th>
<th>$r = 1/4$</th>
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<td>$\pi_S(x)$</td>
<td>$\pi_T(x)$</td>
<td>$\pi_S(x)$</td>
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<td>0</td>
<td>481.25</td>
<td>481.25</td>
<td>481.25</td>
</tr>
<tr>
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<td>125.27</td>
<td>125.53</td>
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<td>59.60</td>
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<tr>
<td>800</td>
<td>0.03</td>
<td>46.48</td>
<td>46.73</td>
</tr>
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<tr>
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</tr>
</tbody>
</table>

exactly by applying a standard fast Fourier transform algorithm to compute $F_T(\theta)$ for all $\theta \in \{1, \ldots, \theta^*\}$ for some sufficiently large value of $\theta^*$. A similar numerical procedure was used to compute $F_T$ as defined in (19) in terms of the conditional Poisson approximations $T(\theta)$ with $E(T) = E(S)$, as detailed in Section 3.2.

As an illustration, Table 1 shows the exact and approximate values of the stop-loss premium for this portfolio when $q = 3$, $b = 2$, and Kendall’s tau is successively equal to 0, 1/8 and 1/4. Two observations are apparent from this table, in which of course $\pi_S(0) = \pi_T(0) = E(S) = E(T) = 481.25$ and $\pi_T(x) \geq \pi_S(x)$ for all $x \geq 0$. First, the approximation is seen to be quite satisfactory for all values of $\tau$ and $x$; the difference $|\pi_S(x) - \pi_T(x)|$ is always smaller than 1, and quite often much smaller than this. Second, it appears that for each fixed value of $x \geq 0$, $\pi_S(x)$ is a monotone increasing function of $\gamma$, so that larger degrees of dependence among the risks translates into a greater stop-loss premium for any given retention amount $x$. This empirical finding may actually be confirmed using the following extension of Proposition 3 in Denuit et al. (2002).

Proposition 4.1. Let $(M_1, \ldots, M_n)$ be a vector of integer-valued random variables. Given sequences $(Y_{1k}), \ldots, (Y_{nk})$ of mutually independent, positive random variables, define the random sum

$$ W_{M_k} = \begin{cases} \sum_{\ell=1}^{M_k} Y_{k\ell} & \text{if } M_k > 0, \\ 0 & \text{if } M_k = 0, \end{cases} $$

for each $k \in \{1, \ldots, n\}$. Suppose that the $M_k$’s are joined by a multivariate distribution whose underlying copula is of the form (9) with generator $\phi = \phi_\gamma$ depending on a parameter $\gamma$. Let also

$$ W = \sum_{k=1}^n W_{M_k}. $$
If $\phi \circ \phi^{-1}$ is a Laplace transform for some possible parameter values $\gamma$ and $\gamma'$, then

$$(M_1, \ldots, M_n) \prec_{sm} (M'_1, \ldots, M'_n)$$

in the supermodular ordering, as defined in Shaked and Shanthikumar (1997), Bäuerle and Müller (1998), or Denuit et al. (2002). As a consequence,

$$(W_{M_1}, \ldots, W_{M_n}) \prec_{sm} (W'_{M_1}, \ldots, W'_{M_n})$$

and hence

$$W \prec_{cx} W'$$

in the classical stop-loss or convex ordering defined in the above references.

Proof. Relation (20) is a direct consequence of Theorem 3.1 of Wei and Hu (2002). Relation (21) then follows from part (iv) of Proposition 2 of Denuit et al. (2002), and finally, part (i) of Corollary 2 in the latter paper implies that $W \prec_{cx} W'$, as claimed.

Corollary 4.1. Suppose that in the setting of Proposition 4.1, the underlying copula of the $M_i$’s belongs to the Clayton, Gumbel or Frank families, as defined in Section 2.2. If $\phi$ represents a generator for any one of these three families, then $\phi \circ \phi^{-1}$ is a Laplace transform whenever $\gamma < \gamma'$, and hence relations (20)–(22) obtain.

Proof. This is a straightforward consequence of results stated in Appendix A.1 of Joe (1997).

Corollary 4.2. Consider a single-class model consisting of $n$ risks of the form $X_\ell = I_\ell B_\ell$, where the $B_\ell$’s are positive random variables that are mutually independent among themselves and of the $I_\ell$’s. Suppose that the latter are Bernoulli random variables whose joint distribution is of the form (8) with Archimedean copula $C$ from the Clayton, Gumbel or Frank families, as parameterized in Section 2.2. The stop-loss premium of the total claim amount $S = X_1 + \cdots + X_n$ is then a monotone increasing function of the dependence parameter $\gamma$, in other words,

$$\gamma < \gamma' \Rightarrow \pi_S(x) \leq \pi_S'(x), \quad x \geq 0.$$

Proof. It suffices to set $I_\ell = M_\ell$ and $B_\ell = Y_\ell 1$ for $1 \leq \ell \leq n$.

To investigate in greater depth the impact of heterogeneity and the strength of dependence on the quality of the compound Poisson approximations, the observed Kolmogorov distance

$$d_q(q, b) = \sup_{x \in \mathbb{R}}|F_q(x) - F_b(x)|$$

between $F_q$ and its approximation $F_b$ was plotted as a function of $q$ and $b$ in Figs. 1–3 for the three values of $\gamma$ corresponding to $\tau = 0$ (independence), $\tau = 1/8$ (small dependence), and $\tau = 1/4$ (moderate dependence). Three main observations emerge from these pictures:

(i) In all graphics, the influence of $b$ seems to be much smaller than that of $q$; that is, greater heterogeneity in the occurrence probabilities is more detrimental to the approximation than heterogeneity in the average amounts of the claims.

(ii) The approximation tends to improve as $\gamma$ increases; in other words, the compound Poisson approximation performs better when the occurrence random variables are dependent than when they are not; nevertheless, the approximation remains quite good even in the case of independence, since $d_q(q, b) \leq 0.05$ on the entire domain.
Fig. 1. Graph of the Kolmogorov distance $d_\gamma(q, b)$ between $F_S$ and its compound Poisson-based approximation $F_T$ for a single-class portfolio comprising 1000 heterogeneous contracts whose occurrence random variables are independent.

Fig. 2. Graph of the Kolmogorov distance $d_\gamma(q, b)$ between $F_S$ and its compound Poisson-based approximation $F_T$ for a single-class portfolio comprising 1000 heterogeneous contracts whose occurrence random variables are stochastically related through a Frank copula model with $\tau = 1/8$. 
(iii) The approximation tends to deteriorate as \( q \) increases; in other words, the compound Poisson approximation performs better when the occurrence random variables are homogeneous than when they are not. Indeed, the approximation is best when \( q = 0 \), even in the presence of heterogeneity in the means of the claim amounts \( B_{\ell} \).

The same phenomena, which were verified experimentally using other portfolio compositions and copula dependence structures, are generally in accordance with intuition, except perhaps as concerns item (ii). Theoretical arguments in support of these conclusions would be well worth developing.

Acknowledgements

Partial funding in support of this work was provided by the Natural Sciences and Engineering Research Council of Canada, by the Fonds québécois de recherche sur la nature et les technologies, and by the Chaire en assurance L’Industrielle—Alliance (Université Laval). The authors are grateful to Jean-François Plante for some programming assistance.

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